Mathematics

For Rwanda Schools

Senior 3

Student's Book

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Contents

1 PROBLEMS ON SETS

Key unit competence: By the end of this unit, the learner should be able to solve problems involving sets.

Unit outline

- Analysis and interpretation of problems using sets.
- Representation of a problem using a Venn diagram.

Introduction

Unit focus Activity

Use your knowledge on sets to solve the following problem.

"For planning purposes, a Physical Education (P.E) teacher asked a Senior 3 class of 24 students to vote by raising of hands for the ball games they liked playing from among football, volleyball and basketball. After the voting, he observed that each of the 24 students liked **at least** one game. 1 student liked all the three games. 2 students liked volleyball and basketball but not football. 2 students liked volleyball and football but not basketball. In summary, he noted that 6 students liked volleyball, 12 liked basketball and 15 liked football".

The teacher went back to the staffroom and realised that he had not established the number of students who liked football and basketball but not volleyball. He has called you to help him determine that number using your knowledge, to avoid calling the whole class to vote again. Kindly, determine the number and give to the teacher.

The knowledge of operations on set is very useful in solving some complex real life problems that are not easy to solve through other analytical methods. In Senior 1 and Senior 2, we learned some basic concepts and operations on sets. In this unit, we will practice the application of those concepts and extend them to solve slightly more challenging problems.

1.1 Review of union, intersection and complement of sets

Activity 1.1

1. Remind your partners what a set is.
2. Given that A = {East African community countries}
   B = {countries bordering Rwanda}
   C = {countries which share Lake Victoria}:
   (a) With the aid of a map or an atlas, list the members of:
   (i) Set A (ii) Set B
   (iii) Set C
   (b) Find: (i) $n(A)$ (ii) $n(B)$
   (iii) $n(C)$
   (c) Find: (i) $(A\cap B)$ (ii) $(A\cap C)$
   (iii) $(A\cup B\cup C)$
   (iv) $(A\cup B\cap C)$
3. Given that $\varepsilon = \{1, 2, 3, 4, 5, 6\}$; $A = \{2, 3, 5\}$; $B = \{3, 4, 5\}$, list the members of:
   (i) A' (ii) B'
   (iii) $A'\cap B$ (iv) $A'\cup B'$
   (v) $(A\cap B)'$ (vi) $A'\cap B'$

The set of common elements which appear in two or more sets is called **the intersection of the sets**. The symbol used to denote intersection of sets is $\cap$.

Intersection of sets is also represented by **"and"** in word statement. For example, "sets A and B" means $A \cap B$.

When the elements of two or more sets are put together to form a set, the set formed is known as **union of sets**. The symbol for the union of sets is $\cup$.

Union of sets is also represented by **"or"** in word statement. For example, "Sets A or B" means $A \cup B$ that is the union of sets A and B.

Complement of a set is the set of all elements in the universal set that are not members of a given set. The complement of set **A** is denoted by **A**'.

A **universal set** contains all the subsets under consideration. It is denoted by the symbol ε.

Example 1.1

Given the following sets $A = \{a, b, c, d, e, f\}$ *and* $B = \{a, b, c, h, i,\}$ *find:*

(i) $(A \cap B)$ *(ii)* $(A \cup B)$

Solution

(i) $(A \cap B) = \{a, b, c\}$

(ii) $A \cup B = \{a, b, c, d, e, f\} \cup \{a, b, c, h, i,\}$

$= \{a, b, c, d, e, f, h, i\}$

Example 1.2

Given $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6\}$ *and* $C = \{1, 3, 5, 7, 9\}$, *answer the questions below about the sets A, B and C.*

(a) List the set $A \cap B$.

(b) Write down $n(A)$.

(c) List the set $A \cup B$.

(d) List the set $A \cup B \cup C$.

Solution

(a) $A \cap B = \{2, 4\}$

(b) $n(A) = 5$

(c) $A \cup B = \{1, 2, 3, 4, 5, 6\}$

(d) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 9\}$

Exercise 1.1

1. If $A = \{2, 4, 6, 8\}$, $B = \{1, 2, 3\}$, $C = \{6, 8, 10\}$ and $D = \{2, 3, 6\}$ find:

(a) $n(A)$ (b) $n(B)$ (c) $n(C)$

(d) $n(A) + n(B)$

(e) $n(A) + n(C) - n(B)$

(f) $A \cup B \cup C$

(g) $n(A \cup B \cup C)$

(h) $n(A \cup C \cup D)$

2. Find the union of the following sets:

A = {positive even numbers from 0 to 20}

B = {Integers greater than -2 but less than 9}

C = {Prime numbers between 1 and 7}

3. If P = {counting numbers from 1 to 15} and Q = {Even numbers from 2 to 14}, find:

(i) $(P \cup Q)'$ (ii) $(P \cap Q)'$

(iii) ε (iv) n(ε)

4. Given that $A = \{3, 5\}$, $B = \{7, 9, 11, 13\}$, $C = \{3, 5, 7\}$ and ε = {3, 5, 7, 9, 11, 13}, find:

(i) A' (ii) $(A \cap B)'$ (iii) B'

(iv) $(A \cup B)$ (v) $(A \cap C)'$

(vi) $(B \cap C)'$

1.2 Representation of problems using a Venn diagram

1.2.1 Venn diagrams involving two sets

Activity 1.2

A survey was carried out in a shop to find the number of customers who bought bread or milk or both or neither. Out of a total of 79 customers for the day, 52 bought milk, 32 bought bread and 15 bought neither.

(a) Without using a Venn diagram, find the number of customers who:

(i) bought bread and milk

(ii) bought bread only

(iii) bought milk only

(b) With the aid of a Venn diagram, work out questions (i), (ii) and (iii) in (a) above.

(c) Which of the methods in (a) and (b) above is easier to work with? Give reasons for your answer.

From the activity above, we clearly see that a Venn diagram plays a very important role in analysing the set problem and helps in solving the problem very easily.

First, express the data in terms of set notations and then fill the data in the Venn diagram for easy solution.

Some important facts like "intersection", "union" and "complement" should be well considered and represented when drawing Venn diagrams.

Consider two intersecting sets A and B such that A = {a, b, c, d, e, f} and

B = {a, b, c, d, g, i, j, k, l}.

We represent the two sets in a set diagram as shown in *Fig 1.1* below.

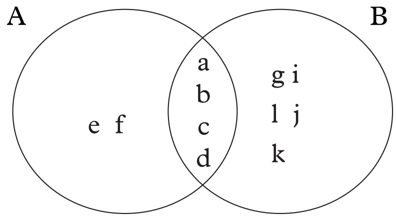


Fig 1.1

The union of sets A and B is given by the number of elements.

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

In the Venn diagram in *Fig 1.1,*

$n(A) = 6$, $n(B) = 9$ *and* $n(A \cap B) = 4$

$\Rightarrow n(A \cup B) = 6 + 9 - 4 = 11$

Example 1.3

Fig. 1.2 shows the marks out of 15 scored by a number of Senior 3 students in groups C and D.

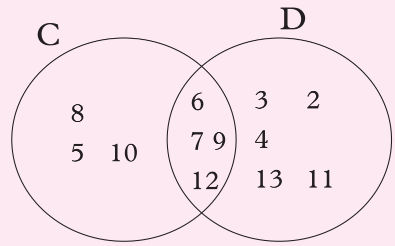


Fig 1.2

Determine the total number of Senior 3 students in the two groups.

Solution

$n(C \cup D) = n(C) + n(D) - n(C \cap D)$

$= 7 + 9 - 4$

$= 16 - 4$

$= 12$

Example 1.4

A survey involving 120 people about their preferred breakfast showed that;

55 drink milk at breakfast,

40 drink juice at breakfast and

25 drink both milk and juice at breakfast.

(a) Represent the information on a Venn diagram.

(b) Calculate the following:

(i) Number of people who take milk only.

(ii) Number of people who take neither milk nor juice.

Solution

(a) Let A be the set of those who drink milk and B be the set of those who drink juice, x be the number of those who drink milk only, y be the number of those who drink juice only and z represents number of those who did not take any.

By expressing data in set notation;

$n(A) = 55, n(B) = 40, n(A \cap B') = x,$
$n(A \cap B) = 25, n(A' \cap B) = y.$

$n(\varepsilon) = 120.$

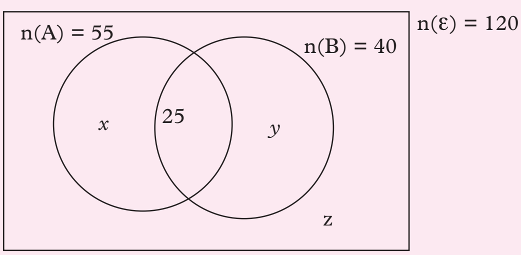


Fig. 1.3

(b) (i) We are required to find the number of those who take milk only.

$x = 55 - 25 = 30$

So, 30 people take milk only.

(ii) To find the value of z;

$30 + 25 + 15 + z = 120.$

$z = 120 - (30 + 15 + 25).$

$z = 120 - 70 \Rightarrow z = 50.$

So, 50 people take neither eggs nor juice for breakfast.

Example 1.5

In a class of 20 pupils, 12 take Art (A) and 10 take Chemistry (C). The number that take none is half the number that take both.

(a) Represent the information on a Venn diagram.

(b) Use the Venn diagram to determine the number that take:

(i) Both (ii) None

Solution

We first extract the data and represent it in set notation.

$n(\varepsilon) = 20, n(A) = 12, n(C) = 10$

Let $n(A \cap C) = y$ so, $n(A \cup C)' = \frac{1}{2}y$

(a)

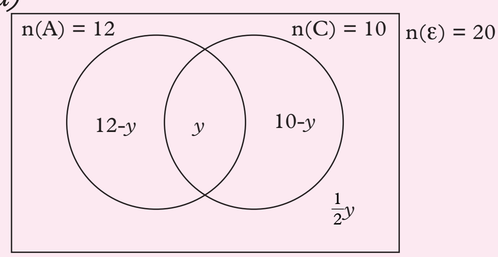


Fig. 1.4

(b) By solving for the value of y;

$12 - y + y + 10 - y + \frac{1}{2}y = 20$

So, $22 - \frac{1}{2}y = 20$

Collecting like terms together,

we get, $-\frac{1}{2}y = 20 - 22$

$-\frac{1}{2}y = -2$

$-y = -4$

$y = 4$

(i) Those who take both are equal to 4.

(ii) Those who take none

$= \frac{1}{2} \times 4 = 2$

Exercise 1.2

1. In a certain group of children, all of them study French or German or both languages. 15 study French but not Germany, 12 study German of whom 5 study both languages.
   (a) Draw a Venn diagram to show this information.
   (b) Calculate how many children are there in the group.
2. In a class of 30 students, students are required to take part in at least one sport chosen from football and volleyball. 18 play volleyball, 22 play football. Some play the two sports.
   (a) Draw a Venn diagram to show this information.
   (b) Use your diagram to help determine the number of students who play the two sports.
3. In a group of 17 pupils, 10 offer Economics and 9 offer Mathematics. The number that offer both Economics and Mathematics is twice the number that offer none of the two subjects.
   (a) Draw a Venn diagram to represent the information.
   (b) Calculate the number of pupils that;
      (i) Offer both subjects
      (ii) Offer only one subject
      (iii) Offer none of the subjects.
4. Of 35 students in a class, 26 play football, 20 play volleyball and 17 play both games.
   (a) Represent the information on a Venn diagram.
   (b) Calculate the number of students who play neither of the games.
5. In a school of 232 students, 70 are members of Anti-AIDS club, 30 are members of debating club and 142 do not belong to any of the mentioned clubs.
   (a) Represent the information on the Venn diagram.
   (b) Use the Venn diagram to calculate the number of students who belong to debating club only.

DID YOU KNOW?

Being a member of Anti-AIDS club can help you to learn many methods of keeping yourself safe and free from HIV-AIDS. Being a member of debating club can also help you to become a good public speaker.

6. The pupils of senior three class were asked about the sports they play. 17 of them play football. 14 play tennis. 5 of them play both football and tennis. There are 30 pupils in the class.
   (a) Draw a Venn diagram to show this information.
   (b) How many play football but not tennis?
   (c) How many play neither football nor tennis?
7. For the two events A and B, we are given that, $n(A \cap B) = 5$, $n(A) = 11$, $n(A \cup B) = 12$ and $n(B') = 8$.
   (a) Copy and complete the Venn diagram below.

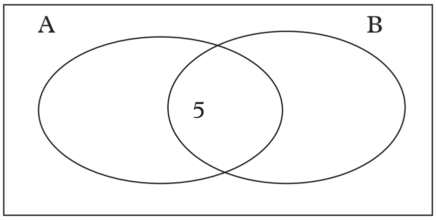


Fig. 1.5

(b) Find:

(i) $n(A \cap B)'$

(ii) $n(A')$

8. A group of 50 married men were asked if they gave their wives flowers or chocolates on Valentine's Day. Results revealed that 31 gave chocolates, 12 gave flowers and 5 gave both flowers and chocolates.

(a) Represent the information on a Venn diagram.

(b) Find the number of men who;

(i) Gave flowers only.

(ii) Gave Chocolates only.

(iii) Gave neither flowers nor chocolates.

1.2.2 Venn diagrams involving three sets

Activity 1.3

A survey was done on 50 people about which food they like among rice, sweet potatoes and posho. It was found out that 15 people like rice, 30 people like sweet potatoes, 19 people like posho. 8 people like rice and sweet potatoes, 12 rice and posho, 7 people like Sweet potatoes and posho. 5 people like all the three types of food.

(a) Extract the data and represent it in set notation.

(b) Without using a Venn diagram;

(i) Find the number of people who like none of the foods.

(ii) Find the number of those who like posho and rice only.

(iii) Find the number of those who like sweet potatoes and rice only.

(c) With the help of a Venn diagram find out the solutions for (b) (i), (ii) and (iii) above.

(d) Was it easy to do (b) (i), (ii) and (iii) without a Venn diagram?

From the above activity, it is clearly seen that without a Venn diagram, the problems involving three or more sets become complicated to handle.

A Venn diagram makes the problem easier because we can represent the data extracted in each region and then calculate the values required.

Consider the venn diagram shown in *Fig. 1.6* showing the numbers of students who take the foreign languages German (G), Spanish(S) and French(F) in a college.

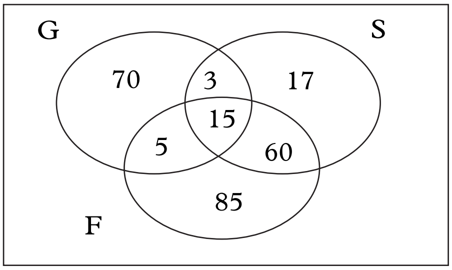


Fig. 1.6

The total number of students taking languages is given by the union of the three sets as shown by the following formula.

$n(G \cup S \cup F) = n(G) + n(S) + n(F)$

$- \{n(G \cap S) + n(G \cap F) + (S \cap F) + n(G \cap S \cap F)\}$

From the Venn diagram in *Fig 1.6*,

$n(G \cup S \cup F) = 93 + 95 + 165$

$- (18 + 20 + 20 + 75) + 15$

$= 353 - 113 + 15$

$= 255$

Example 1.6

The students in Senior 3 class did a survey on their names regarding whether they contained the letters B, C and D. The following Venn diagram shows the results of the survey in terms of the number of names in each category:

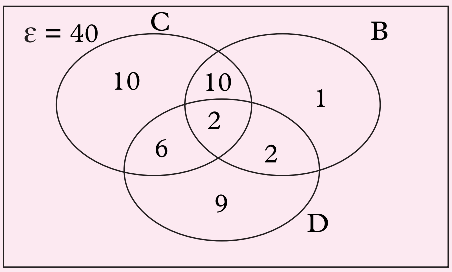


Fig. 1.7

Use the Venn diagram to determine the number student's names that contained in;

(a) *All the three letters*

(b) *Letter D*

(c) *Letters B and D but not C*

(d) *Only two of the letters*

(e) *The total number of students*

Solution

(a) *All the three letters*

$n(B \cap C \cap D) = 2$

(b) $n(D) = 9 + 6 + 2 + 2 = 19$

(c) *Letters B and D but not C*

$n(B \cap D) - n(B \cap C \cap D) = 4 - 2 = 2$

(d) *Only two of the letters*

$= 6 + 10 + 2 = 18$

(e) *The total number of students*

$n(B \cup C \cup D) = n(B) + n(C) + (C) - \{n(B \cap C) + n(B \cap D) + n(C \cap D)\} + n(B \cap C \cap D)$

$= 15 + 28 + 19 - \{12 + 4 + 8\} + 2$

$= 62 - 24 + 2$

$= 40$ *Students*

Example 1.7

A group of 40 tourists arrived in Rwanda and visited Akagera National park (A), Nyungwe forests (N) and Virunga mountains (V). Results showed that 33 visited Akagera, 21 visited Nyungwe and 23 visited Virunga. 18 visited both Akagera and Nyungwe, 10 visited both Nyungwe and Virunga, and 17 visited both Akagera and Virunga. All tourists visited at least one of the places.

(a) *Represent the information on a Venn diagram.*

(b) *Find the number of tourists that visited:*

(i) *Akagera only.*

(ii) *Did not visit Nyungwe.*

Solution

$n(\varepsilon) = 40.$

$n(A) = 33, n(N) = 21, n(V) = 23.$

$n(A \cap N) = 18, n(N \cap V) = 10,$

$n(A \cap V) = 17.$

Let $n(A \cap N \cap V) = y$

(a)

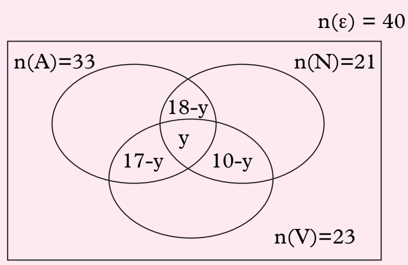


Fig. 1.8

$n(A)$ *only* $= n(A \cap N' \cap V').$

$n(A \cap N' \cap V') = 33 - (18 - y + y + 17 - y).$

$n(A \cap N' \cap V') = 33 - 35 + y = y - 2.$

$n(A \cap N' \cap V') = y - 2.$

$n(N)$ *only* $= n(A' \cap N \cap V').$

$n(A' \cap N \cap V') = 21 - (18 - y + y + 10 - y).$

$n(A' \cap N \cap V') = 21 - 28 + y = -7 + y.$

$n(A' \cap N \cap V') = y - 7.$

$n(V)$ only $= n(A' \cap N' \cap V).$

$n(A' \cap N' \cap V) = 23 - (17 - y + y + 10 - y).$

$n(A' \cap N' \cap V) = 23 - 27 + y.$

$n(A' \cap N' \cap V) = y - 4.$

The Venn diagram in Fig. 1.9 shows the data in specific regions.

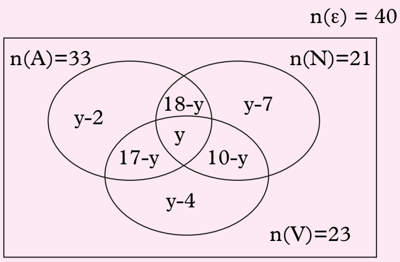


Fig. 1.9

$y - 2 + 18 - y + y - 7 + 17 - y + y + 10 - y + y - 4 = 40$

$y + 32 = 40.$

$y = 8.$

(b) (i) *Those who visited Akagera only are* $y - 2 = 8 - 2 = 6.$

(ii) *Those who did not visit Nyungwe are* $y - 4 + 17 - y + y - 2$

$= 8 - 4 + 17 - 8 + 8 - 2$

$= 19$

Example 1.8

The Venn diagram below shows the allocation of the members of the Board of Directors of a school in three different committees; Academic (A), Production (P) and Finance (F).

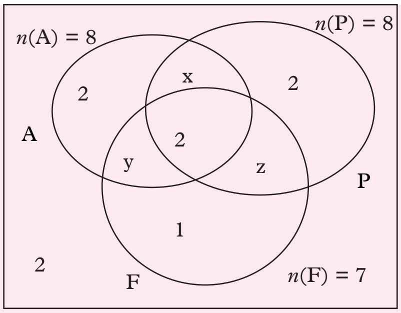


Fig. 1.10

(a) *Determine the values of x, y and z.*

(b) *What is the total number of members in the Board of Directors?*

(c) *Find the number of those who are not members of the academic committee.*

(d) *How many belong to at least two committees?*

Solution

(a) *For Academic,*

$2 + x + 2 + y = 8.$

$x + y = 8 - 4.$

$x + y = 4 \ldots\ldots\ldots (i)$

For Production,

$2 + x + 2 + z = 8.$

$x + z = 8 - 4.$

$x + z = 4 \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

For Finance,

$1 + y + 2 + z = 7.$

$y + z = 7 - 3.$

$y + z = 4 \ldots\ldots\ldots\ldots\ldots (iii)$

Make x the subject in equation (i)

$x = 4 - y \ldots\ldots\ldots (iv)$

Substitute equation (iv) into (ii)

$4 - y + z = 4.$

$-y + z = 0 \ldots\ldots\ldots (v)$

Solving (iii) and (v) simultaneously

$$\begin{array}{r} y + z = 4 \\ + \quad -y + z = 0 \end{array}$$

$2z = 4 \Rightarrow z = 2$

So, $y = 4 - z = 4 - 2 = 2$

$\therefore y = 2.$

$x + y = 4$ and so $x = 4 - y = 4 - 2 = 2.$

$\therefore x = 2.$

(b) *Total number of members are*

$2 + 2 + 2 + 2 + 2 + 2 + 1 + 2 = 15$

(c) *Those who are not members of academic committee are:* $2 + 1 + z + 2$

$= 2 + 1 + 2 + 2 = 7$

(d) *Those who belong to atleast two committees are:* $y + 2 + x + z$

$= 2 + 2 + 2 + 2 = 8$

Exercise 1.3

1. In a class of 53 students, 30 study Chemistry, 20 study Physics, 15 study Mathematics. 6 study both Chemistry and Physics, 4 study both mathematics and Chemistry, 5 study both Physics and Mathematics. All the students study at least one of the subjects.
   (a) Represent the information on a Venn diagram.
   (b) Find the number of students who study all the three subjects.
   (c) How many study;
      (i) Physics only.
      (ii) Physics but not Mathematics
      (iii) Two subjects only.
2. Out of 100 students in a school, 42 take English, 35 take Kiswahili and 30 take French. 20 take none of the subjects, 9 take French and English, 10 take French and Kiswahili, 11 take English and Kiswahili.
   (a) Represent the information on a Venn diagram.
   (b) Find the number of students who take three subjects.
   (c) Find the number of students who take English only.
   (d) Find the number of students who take Kiswahili and French.
3. A school has a teaching staff of 22 teachers. 8 of them teach Mathematics, 7 teach Physics and 4 teach Chemistry. 3 teach both Mathematics and Physics, none teaches Mathematics and Chemistry. No teacher teaches all the three subjects. The number of teachers who teach Physics and Chemistry is equal to the number of teachers who teach Chemistry but not Physics.
   (a) Represent the data on a Venn diagram.
   (b) Find the number of teachers who teach;
      (i) Mathematics only.
      (ii) Physics only.
      (iii) None of the three subjects.
4. In a class of 60 students, 15 are members of debating club (D), 30 are members of never again club (A) and 20 are members of Science club (S). 3 are members of debating and never again only. 4 are members of never again and science club only while 1 is a member of debating and science club only. 7 students do not belong to any of the clubs.
   (a) Represent the data on the Venn diagram.
   (b) Find the number of students that belong to;
      (i) Only one club.
      (ii) Atleast two clubs.

(iii) Do not belong to debating club.

5. In a class of 45 students, 7 like Mathematics (M) only, 2 like Physics (P) only, and 3 like Chemistry (C) only. 18 like Mathematics and Physics, 16 like Physics and Chemistry, 14 like Mathematics and Chemistry. The number of students who like none of the three subjects is half the number of those who like all the three subjects.

(a) Show the above information in a Venn diagram.

(b) Determine the number of students who;

(i) Like none of the three subjects.

(ii) Who do not like Mathematics.

6. A survey involving 50 people was done to find out which religious events they attend among Catholics, Protestants, and Muslim. It was found out that 15 people attend Catholic event, 30 people attend Protestant event, 19 people attend Muslim event. 8 people attend both Catholic and protestant events, 12 people attend both Catholics and Muslim events, 7 people attend both Protestant and Muslim events. 5 people attend all the three categories of religious events.

(a) Represent the information on the Venn diagram.

(b) (i) How many people attend Catholic event only?

(ii) How many attend Catholic and Protestant events, but not at Muslim event?

(iii) How many people do not attend any of these religious events?

BEWARE!!!

Religious differences should not cause divisionism. We should all learn to value one another to stay together as peaceful Rwandans.

Unit Summary

- A *set* - is a well-defined collection of distinct objects, considered as an object in its own right.

  For example, the numbers 2, 4, and 6 are distinct objects when considered separately, but when they are considered collectively, they form a single *set* of size three, written $\{2, 4, 6\}$.

- Union of a set: In set theory, the union (denoted by $\cup$) of a collection of sets is the set of all elements in the collection. It is one of the fundamental operations through which sets can be combined and related to each other.

  For example if $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8, 9\}$, then we have a union set for A and B as:

  $A\cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- Intersection of sets: In mathematics, the intersection $A\cap B$ of two sets A and B is the set that contains all elements of A that also belong to B (or equivalently, all elements of B that also belong to A), but no other elements. For three sets A, B and C, we write $A\cap B\cap C$.

  For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8, 9\}$, then we have an intersection set for A and B as $A \cap B = \{4, 5\}$

- Compliment of a set. In set theory, the complement of a set A refers to elements not in A. The relative complement of A with respect to a set B, also termed the difference of sets A and B, written as B \ A, is the set of elements in B but not in A.

  For example, consider universal set U = {1, 2, 3, 4, 5, 6, 7}, and we will define our subset as A = {1, 3, 4}. The complement of A is the set of all the elements in U that are not in A. Therefore, the complement of A is {2, 5, 6, 7}.

- A Venn diagram. It is a diagram representing mathematical or logical sets pictorially as circles or closed curves within an enclosing rectangle (the universal set), common elements of the sets being represented by intersections of the circles.

Unit 1 Test

1. The following facts were discovered in a survey of course preferences of 110 pupils in senior six: 21 like engineering only, 63 like engineering, 55 like medicine and 34 like none of the two courses.
   (a) Draw a Venn diagram representing this information.
   (b) (i) How many like Engineering or Medicine?
   (ii) How many like Engineering and Medicine?
   (iii) How many like only Medicine?

2. A survey was carried out in a shop to find out the number of customers who bought bread or milk or both or neither. Out of a total of 79 customers for the day, 52 bought milk, 32 bought bread and 15 bought neither.
   (a) Draw a Venn diagram to show this information and use it to find out:
   (b) (i) How many bought bread and milk.
   (ii) How many bought bread only.
   (iii) How many bought milk only.

3. Five members of Mathematics club conducted a survey among 150 students of Senior 3 about which careers they wish to join among Engineering and Medical related courses. 83 want to join Engineering, 58 want to join medical related courses. 36 do not want to join any of the careers.

   Represent the data on the Venn diagram. Find the number of students who wish to join both careers.

4. A survey was done on 50 people about which hotels they eat from among H, S and L. 15 people eat at hotel H, 30 people eat at hotel S, 19 people eat at hotel L, 8 people eat at hotels H and S, 12 people eat at hotels H and L, 7 people eat at hotels S and L. 5 people eat at hotels H, S and L.
   (a) How many people eat only at Hotel H?
   (b) How many people eat at hotels H and S, but not at L?
   (c) How many people don't eat at any of these three hotels?

5. A survey involving 50 students was carried out and research revealed that 21 of them like Kiswahili (K) while 32 of them like Mathematics (M).
   (a) Represent the information in the Venn diagram.
   (b) How many students like only one subject?

6. A group of 50 people were asked about the sections they read very keenly in a newspaper among politics, advertisements and sports. The results showed that 25 read politics, 16 read advertisement, 14 read sports. 5 read both politics and advertisement, 4 read both advertisement and sports, 6 read both politics and sports, and 2 read all the three sections.
   (a) Represent the data on the Venn diagram.
   (b) Find the number of people who read;
      (i) At least one of the three sections.
      (ii) Only one of the three sections.
      (iii) Only politics.

7. Given that, $n(A \cup B) = 29$, $n(A) = 21$, $n(B) = 17$, $n(A \cap B) = x$.

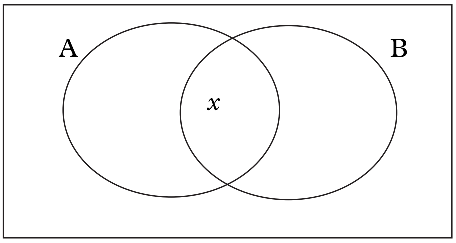


Fig. 1.11

   (a) Write down in terms of the elements of each part.
   (b) Form an equation and hence find the value of x.

8. In a school, each student takes atleast one of these subjects; Mathematics, Physics and Chemistry. In a group of 60 students, 7 take all the subjects, 9 take Physics and Chemistry only, 8 take Physics and Mathematics, 5 take Mathematics and Chemistry only. 11 students take Mathematics only, 2 take Physics only and 15 students take Chemistry only.
   (a) Draw a Venn diagram for the information above.
   (b) Find the number of those who do not take any of the subjects.
   (c) Find the number of students who take Mathematics.

2 NUMBER BASES

Key unit competence

By the end of this unit, the learner should be able to present number bases and solve related problems.

Unit Outline

- Definition of number bases.
- Change of base.
- Operations using bases (addition, subtraction, division and multiplication)
- Special bases (binary and duodecimal systems)
- Solving equations involving different bases.

Introduction

Unit Focus Activity

In everyday life, we count or estimate quantities using groups of ten items or units. This may be so because, naturally, we have ten fingers. For example, when we count ten, i.e. we write 10 meaning one group of 10 and no units. A quantity like twenty five, written as 25 means 2 groups of 10 and 5 units

Suppose instead we had say 6 fingers

(i) How, in your opinion would we do our counting?

(ii) If we had eight fingers, how would we count?

(iii) Demonstrate symbolically how counting in groups of 3, 4, 5, 6, 7… can be done.

(iv) Do you think we could also do operations such as addition, subtraction, multiplication and division using such groups? If your answer is yes, demonstrate this with simple examples.

In this unit, we will learn a number of different numeration systems including the decimal (base ten) system that we are all familiar with. We will also learn how to convert between different numeration (counting) systems.

2.1 Numbers and numerals

Activity 2.1

Use a dictionary or internet to define:

(i) Number

(ii) Numeral

(iii) Digit

In mathematical numeral systems, we use basic terms such as number, numeral and digit. In order to deal with number bases, we must be able to distinguish between the three terms.

A number is an idea, a numeral is the symbol that represents the number. The number system that we use today is a place value system. A unique feature of this system is that the value of any of the digits in a number depends on its position.

For example the number 7 707 contains three sevens, and each of them has a particular value as shown in table 2.1.

Thousands	Hundreds	tens	Ones
7	7	0	7

Table 2.1

The 1st seven from the right represents 7 ones or units. The 2nd seven stands for 100s or 10^2 and the 3rd seven stands for 1 000s or 10^3.

The zero holds the place for the tens (10s) without which, the number would be 777 which is completely different from 7 707.

A digit is any numeral from 0 to 9. A numeral is made of one or more digits. For example, number one hundred and thirty five is represented by the numeral 135 which has three digits 1, 3 and 5.

The number 7 707 contains four digits, each of which has a specific value depending on its place value.

The abacus

Activity 2.2

1. Use a Mathematics dictionary or internet, to describe an abacus.
2. Describe how the abacus is used to count in base ten.

One device that has been used over time to study the counting in different numeration systems is the abacus.

An abacus is a calculating device consisting of beads or balls strung on wires or rods set in a frame. Fig. 2.1, shows a typical abacus on which the place value concept can be developed very effectively.

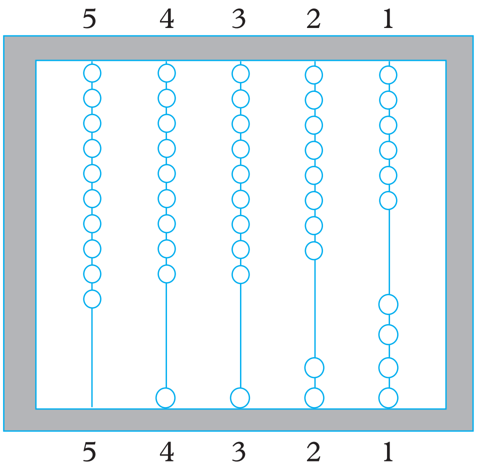


Fig. 2.1

On each wire, there are ten beads. Let us consider the beads at the bottom of the wire. Beginning from the right:

10 beads on wire 1 can be represented by 1 bead on wire 2. Similarly, 10 beads on wire 2 can be represented by 1 bead on wire 3 and so on.

This means:

1 bead in wire 1 represents a single bead.

1 bead in wire 2 represents 10 beads.

1 bead in wire 3 represents (10×10) beads.

1 bead in wire 4 represents $(10 \times 10 \times 10)$ beads.

So, the number shown in Fig. 2.1 is 1 124.

If we had x beads in each wire such that $x < 10$, it would mean that:

In wire 1 we had x beads

In wire 2 we had $10x$ beads

In wire 3 we had 10^2x beads

In wire 4 we had 10^3x beads and so on.

The place values from right to left are

10^0	10^1	10^2	10^3	10^4 ...
Ones	10s	100s	1 000s	10 000s etc

2.2 Number bases

2.2.1 Definition of number bases

Activity 2.3

1. Use a dictionary or internet to find the meaning of number bases.
2. Give some examples of number bases.

Why do you think we count in groups of ten?

If we had 6 fingers, most probably we would count using groups of 6, if 8 fingers, groups of 8 and so on. In the system that we use, every **ten** items make **one** basic group which is represented in the next place value column to the left as shown in Fig. 2.2 below.

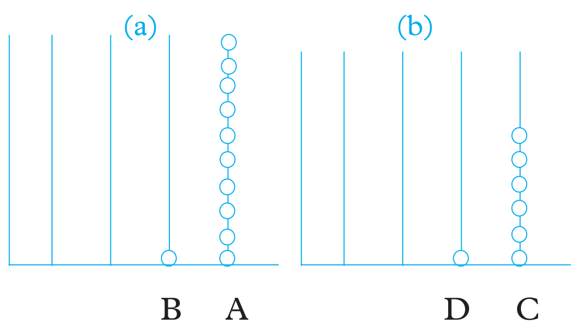


Fig. 2.2

(a) The 1 bead in wire B represents 10 beads in wire A i.e. it represents a group of 10 beads.

(b) The 1 bead in wire D represents 6 beads in wire C, thus making a group of 6 beads.

Counting in different groups of numbers such as 10, 6, 5, 8 etc means using different number systems. We call them base ten, base six, base five, base eight respectively etc.

Now consider Fig. 2.3.

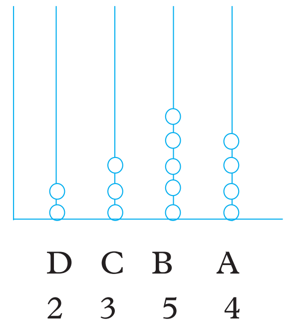


Fig. 2.3

Counting in base six, what numbers do the beads on each wire represent?

(i) There are 4 beads in wire A. This represents 4 ones.

(ii) There are 5 beads in wire B. This means 5 groups of 6 beads each.
i.e. $5 \times 6 = 30$ beads written as 50_{six}.

(iii) There are 3 beads in wire C. This means 3 groups of six sixes i.e. $3 \times 6 \times 6 = 108$ beads, written as 300_{six}.

(iv) There are 2 beads in wire D. This means 2 groups of six six sixes ie
$6 \times 6 \times 6 = 216 \times 2 = 432$
written as $2\ 000_{six}$.

The whole number represented in Fig. 2.3 is $4_{six} + 50_{six} + 300_{six} + 2\ 000_{six} = 2\ 354_{six}$

The answer $2\ 354_{six}$ is read as; two three five four base six. The number $2\ 354_{six}$ has a value of $575_{ten.}$

Example 2.1

Given that the number represented in Fig. 2.4 is in base six, find the number in base 10.

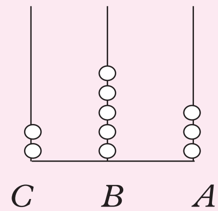


Fig. 2.4

Solution

Column A represents 3 ones.

Column B represents 5 sixes.

Column C represents 2 six sixes.

$\therefore$ *the number* $= (3 \times 6^0) + (5 \times 6) + (2 \times 6^2)$

$= 3 + 30 + 72$

$= 105_{ten}$

$\therefore \quad 253_{six} = 105_{ten}$

Note that 253_{six} *and* 105_{ten} *are two different symbols for the same number.*

2.2.2 Change of base

(a) Changing from base 10 to any other base

Activity 2.4

Consider the number 725 given in base ten.

1. Divide 725 by 8 and write down the remainder.
2. Divide the quotient obtained in (1) above and write down the remainder.
3. Repeat this process of division by 8 until the quotient is less than 8 which you should treat as a remainder and write it down.
4. Write down the number made by the successive remainders beginning with the first one on the right going left.
5. Describe the number in part (4) above in terms of a base.

In this activity, you have just converted 725_{10} to a number in base 8. In converting any number from base ten to any other base, we use successive division of the number by the required base. The new number is obtained by writing down the remainders beginning with the first remainder on the right to the last remainder on the left.

For example, to change 425_{10} to base 6, we do successive division by 6.

$425 \div 6$	=	70	Rem 5
$70 \div 6$	=	11	Rem 4
$11 \div 6$	=	1	Rem 5
$1 \div 6$	=	0	Rem 1

The successive remainders read upwards form the number 1545.

$\therefore \quad 425_{10} = 1\,545_6$

Example 2.2

Convert 92_{10} *to base 6.*

Solution

To convert 92_{10} *to base 6, we perform successive divisions by 6 until the remainder is less than 6.*

6	92	
6	15 Rem 2 ↑	$92 \div 6 = 15$ rem 2
6	2 Rem 3	$15 \div 6 = 2$ rem 3
	0 Rem 2	$2 \div 6 = 0$ rem 2

$92_{10} = 232_6$

Example 2.3

Convert 194_{10} *to base 8*

Solution

194 is divided by 8 successively until the remainder is less than 8.

8	194
8	24 Rem 2 ↑
8	3 Rem 0
	0 Rem 3

$194_{10} = 302_8$

Exercise 2.1

1. Convert the following numbers from base 10 to base 5.
   (a) 50 (b) 36 (c) 231
2. Convert the following numbers in base 10 to base 9.
   (a) 82 (b) 190
   (c) 144 (d) 329
3. Convert the following numbers in base 10 to specified base.
   (a) 145 to base 2
   (b) 5204 to base 6
   (c) 800 to base 2
   (d) 954 to base 8
   (e) 512 to base 3
   (f) 1280 to base 12
   (g) 896 to base 16

(b) Converting any base to base 10

Activity 2.5

Consider the number 125 given in base six.

Using number place value method;

(a) Find the value of digit 1, 2 and 5.

(b) Add up the values obtained in part (a) above.

(c) What does this value represent?

In this activity, you have converted a number from base 6 to base 10. To convert a number from one base to base ten, we use number place values. For example to convert 253_9 to base 10, we say:

235_9 means 5 ones + 3 nines + 2 nine nines.

$\therefore 235_9 = (5\times9^0) + (3\times9^1 + (2\times9^2)$
$= (5 \times 1) + (3 \times 9) + (2 \times 9^2)$
$= 5 + 27 + 162$
$= 194$

$\therefore 235_9 = 194_{10}$

(a) Consider Fig 2.5 below.

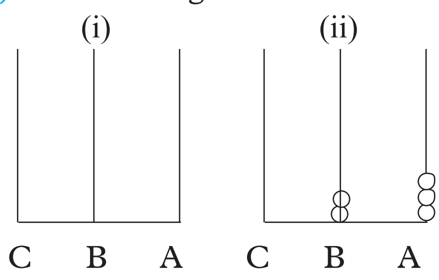


Fig. 2.5

Suppose in Fig 2.5, each spike is designed to hold six beads, and that each bead in spike B represents six beads in spike A. Thus in Fig 2.5 (b) there are two beads in B and three beads in spike A. The 2 beads mean 2 groups of six i.e 2 × 6 or 12 beads. The 3 beads are said to represent 3 ones. Thus the number represented in Fig 2.5(b) is written as 23_{six}

Therefore, $23_{six} = 15_{ten}$

This is read as **two three** base six equal **one five** base ten:

23_{six} and 15_{ten} are different numerals for the same number

(b) Now consider Fig 2.6 below:

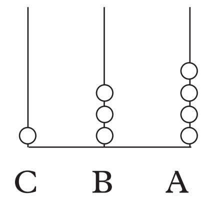


Fig 2.6

The number shown in Fig 2.6 can be written as 134_{six}. What does the single bead in spike C represent? It is the same as six beads in spike B which is equal to six × six (or thirty six) beads in spike A.

Hence 134_{six} means:

The 1 stands for 1 six sixes or 36_{ten}

The 3 stands for 3 six or $3 \times 6_{ten}$

The 4 stands for 4 ones or 4_{ten}

So we would write

134_{six} as $(36 + 18 + 4_{ten}) = 58_{ten}$

i.e $134_{six} = 58_{ten}$

(c) Now consider the number represented in Fig 2.7 below.

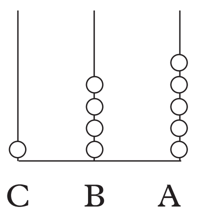


Fig 2.7

When reading off a number in base six, it may help us to think in powers of six. The number represented in Fig 2.7 can be written as

145_{six} = 1×6 sixes + 4 sixes + 5ones

$= (1 \times 6^2) + (4 \times 6) + (5 \times 1)$

$= 36 \quad + \quad 24 \quad + \quad 5$

$= 65_{tens}$

(d) Now, let us think of a number like 28_{ten}. How can we represent 28 on a base six abacus?

We find the number of sixes contained in 28.

To do this we divide 28 by 6. Thus $28 \div 6 = 4$ Rem 4.

So, 28_{ten} is 4 sixes and 4 ones.

This number can be written on the abacus as shown in Fig. 2.8.

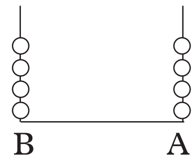


Fig. 2.8

i.e on the abacus, there are 4 beads on spike A and 4 in spike B. i.e

$28_{ten} = 44_{six}$

Use a similar method to show on a base six abacus the following numbers.

81_{ten} and 324_{ten}

Note:

We can use a similar method to represent any base ten number in another base.

Also, a number such as 65_{ten} can be expressed as a number in base six as:

$65 \div 6 = 10$ Rem 5 $\Rightarrow$ 5 ones

$10 \div 6 = 1$ Rem 4 $\Rightarrow$ 4 sixes

$1 \div 6 = 0$ Rem 1 $\Rightarrow 1 \times 6$ sixes

6	65	
6	10 Rem 5	→ ten groups of six and 5 ones
6	1 Rem 4	→ 1 group of six sixes and 4 groups of sixes
	0 Rem 1	

The answer is then written starting with the last remainder, followed by the next remainder, etc vertically up till the first remainder.

$65_{ten} = 145_{six}$

Example 2.4

Express 415_{six} as a number in base ten.

Solution

We use place values to change from base six to base 10.

$415_{six} = (5 \times 1) + (1 \times 6) + (4 \times 6^2)$
$= 5 + 6 + (4 \times 36)$
$= 5 + 6 + 144$
$= 155$
$\therefore 415_{six} = 155_{ten}$

Exercise 2.2

1. Convert the following numbers from specified base to base 10.

(a) 85_9 (b) 1001_2
(c) 2343_5 (d) 12_3
(e) 615_7 (f) 142_5
(g) 1232_4

2. Are the following valid or invalid?

(a) 123_2 (b) 234_5
(c) 1002_2 (d) 3467_6

(c) Converting from one base to any other base

Suppose we wish to change from base m to base n where m ≠ n ≠ 10 and m and n are positive numbers.

Activity 2.6

Consider the number 46_7.

(a) Convert 46_7 to a number in base 10 as you did in activity 2.5
(b) Use your answer to part (a) above and convert it to a number in base 5.
(c) Describe the procedure of converting a number from a number in base x to a number in base y where $x \neq y$.

In this activity, you just converted a number from base seven to base five.

To convert a number from a base other than ten to another base, follow the steps below.

(i) Change or convert the given number to base 10.
(ii) Convert the result of part (i) to a number in the required base, for example,

To convert 121_3 to base 4;

Convert 121_3 to base 10.

Thus $121_3 = 1 + 2 \times 3 + 1 \times 3^2$
$= 16_{10}$

Then convert 16_{10} to base 4, by successive division by 4.

$16 \div 4 = 4$ Rem 0
$4 \div 4 = 1$ Rem 0
$1 \div 4 = 0$ Rem 1

$121_3 = 100_4$

Now let us repeat activity 2.6 using 386_{nine}

- 386_{nine} means 6 ones, 8 nines and 3 nines

We first change 386_{nine} to base ten as follows:

$386_9 = 6 \times 1 + 8 \times 9 + 3 \times 9^2$
$= 6 + 72 + 243$
$= 321_{ten}$

- To convert to base 6, we do successive division of the number in base 10.

Thus,

6	321
6	53 rem 3
6	8 rem 5
6	1 rem 2
6	0 rem 1

$321_{ten} = 1\,253_{six}$

$\therefore 386_9 = 1\,253_6$

Note: to convert a number from a base other than 10 to another base, we first convert from the given base to base 10. Then from base 10 to the required base.

Example 2.5

Convert 514_8 to base 9.

Solution

To convert from base 8 to base a,

(i) First convert to base 10

(ii) Then convert result (i) to base 9

$514_8 = 4 \times 1 + 1 \times 8 + 5 \times 8^2$

$= 4 + 8 + 320$

$= 332_{10}$

To convert 332_{10} to base 9, we do successive division by 9, noting the reminder at each step.

Thus:

9	332	
9	36	Rem 8
9	4	Rem 0
	0	Rem 4

From down upwards the reminders form number 408.

This means 8 ones

0 nines

4 nine-nines

Thus $514_8 = 408_9$.

Exercise 2.3

1. Convert the following to base 7.
   (a) 411_5 (b) 321_6
   (c) 15_6 (d) 302_4
2. Express 63_7 to base 5
3. Given that $85_{10} = 221_x$. Find the value of x.
4. Convert the number 703_8 to;
   (a) Base 6 (b) Base 10
   (c) Base 9 (d) Base 2
   (e) Base 4

In short;

To convert from base ten to another base:

1. Do successive division by the required base noting the remainders at every step.
2. Write down the remainders beginning with the last one on the left.
3. These remainders make up the required number.

To convert from any base x to base 10:

1. Multiply every digit in the number by its place value i.e. 1, x, x^2, x^3 etc.
2. Add the results.

To convert from base m to base n, where m ≠ 10 and n ≠ 10:

1. First convert from base m to base 10:
2. Then, convert from base 10 to base n.

Numbers in other bases can be expressed in the same way as we have done.

The following are some other bases and the numerals used.

Base	Numerals
Nine	0 1 2 3 4 5 6 7 8
Eight	0 1 2 3 4 5 6 7
Seven	0 1 2 3 4 5 6
Six	0 1 2 3 4 5
Five	0 1 2 3 4
Four	0 1 2 3
Three	0 1 2
Two	0 1

and so on.

In any base, the numeral equal to the base is represented by 10.

i.e. $5_5 = 10_5$ $6_6 = 10_6$ $10_{10} = 10$

$8_8 = 10_8$ etc

When a base is greater than 10, say 12, we need to create and define a symbol to represent 10 and 11.

Exercise 2.4

1. Write the first twenty numerals of:
   (a) Base six (b) Base seven
   (c) Base eight
2. What does 8 mean in:
   (a) 108_{ten} (b) 180_{ten}
   (c) 801_{ten} (d) $88\,801_{ten}$
3. Write down in words:
   (a) 203_{six} (b) 302_{four}
   (c) 15_{six} (d) $3\,215_{eight}$
4. Convert the number 703_{eight} to:
   (a) base 6 (b) base 10
   (c) base 9
5. Convert the following into decimal system:
   (a) 411_{five} (b) 321_{six}
   (c) 207_{eight} (d) 750_{nine}
6. Express 63_{seven} to base 5
7. Write in words the meaning of :
   (a) 12_{three} (b) 21_{four},
   (c) 142_{five}, (d) 180_{nine}
8. Use abacus to show place values for the numerals in:
   (a) 211_{five} (b) 615_{seven}
   (c) 173_{eight} (d) $1\,254_{ten}$
9. Convert 118_{nine} to base 5.

2.3 Operations using bases

2.3.1 Addition and subtraction

Activity 2.7

Table 2.2 shows part of the addition table for numerals in a certain base

(i) State the base.

(ii) Copy and complete the table.

(iii) Use your table to evaluate.

15 – 10 13 – 5 12 – 3

+	0	1	2	3	4	5	10
0	0	1			4		10
1				4		10	
2		3			10		
3	3		5			12	
4				11			14
5			11				
10	10						

Table 2.2

Now consider table 2.3.

+	0	1	2	3	4	5	6	7	10
0	0								
1		2							
2			4						
3				6				12	
4					10				
5			11			12			
6							14		
7								16	
10									20

Table 2.3

- Identify the base used in this table.
- Copy and complete the addition

Table 2.3.

- List the numerals used in this table.
- Use your table to formulate some equations involving subtraction.

Note:

To add or subtract, numbers must be in the same base.

In performing addition or subtraction, whatever the base, the digits to be added or subtracted must be in the same place value. For example in $65_{ten} + 18_{ten}$, 5 and 8 have the same place value while 6 and 1 have another place value.

The base used is 8.

This is the required table

+	0	1	2	3	4	5	6	7	10
0	0	1	2	3	4	5	6	7	10
1	1	2	3	4	5	6	7	10	11
2	2	3	4	5	6	7	10	11	12
3	3	4	5	6	7	10	11	12	13
4	4	5	6	7	10	11	12	13	14
5	5	6	7	10	11	12	13	14	15
6	6	7	10	11	12	13	14	15	16
7	7	10	11	12	13	14	15	16	17
10	10	11	12	13	14	15	16	17	20

Table 2.4

- The numerals used range from 0 to 20.
- Some examples of questions and answers

11 – 2 = 7; 17 – 10 = 7,   10 – 1 = 9 etc

Note:

- While working in base eight, eight must not be one of the numerals in use.
- In base eight, there are only 8 digits i.e. 0, 1, 2, 3, 4, 5, 6, 7

Example 2.6

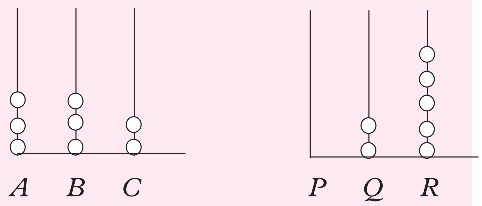
Evaluate: $332_{six} + 25_{six}$

Solution

It is best to set work vertically so that the place values correspond.

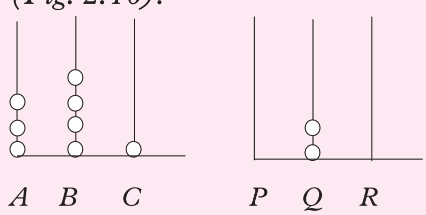
$332_{six} + 25_{six} \Rightarrow 332_6$
$\quad +25_6$

1. *Illustrate the two numbers on different abaci (Fig. 2.9).*



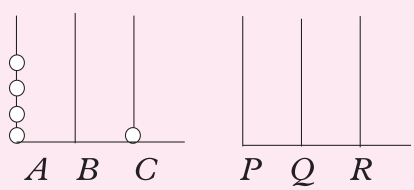
(i) Fig. 2.9 (ii)

2. *Remove all the 5 beads from R and place them in C to make 7 beads. One bead remains at C another goes to B to represent another group of six (Fig. 2.10).*



(i) Fig. 2.10 (ii)

3. *Remove the two beads from Q and place them on B to make 6 beads. No bead remains at B, but one bead goes to A to represent another group of six sixes. (Fig. 2.11).*



(i) Fig. 2.11 (ii)

4. *The result of the addition is* 401_{six}

Alternatively,

$$332_{six} \rightarrow 330 + 2$$
$$25_{six} \rightarrow 20 + 5$$
$$350 + 11$$
$$= 350$$
$$+\ 11$$
$$401_{six}$$

Example 2.7

Use abacus to evaluate:

$52_{eight} - 23_{eight}$

Solution

Fig. 2.12 shows the two numbers on different abaci.

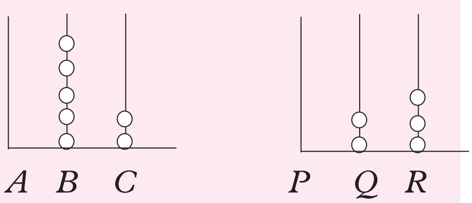


Fig. 2.12

Since we cannot subtract beads in R from beads in C,

1. *Remove one bead from B and place it on wire C so that there is a total of 10 in C, (Fig. 2.13)*

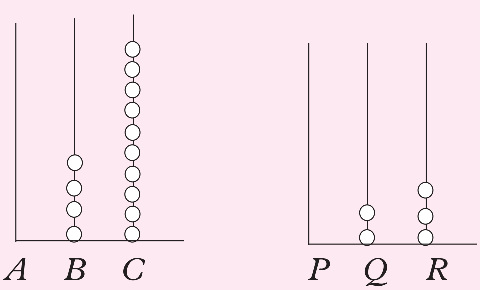


Fig 2.13

2. *Remove 3 beads from C and R (Fig. 2.14).*

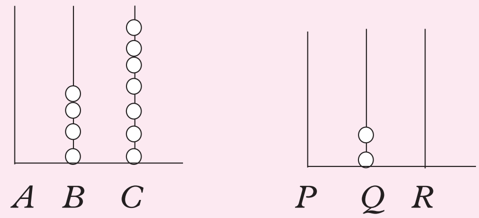


Fig. 2.14

3. *Remove 2 beads from B and Q so that the result is as represented in Fig. 2.14 below.*

   $\therefore\ 52_8 - 23_8 = 27_8$

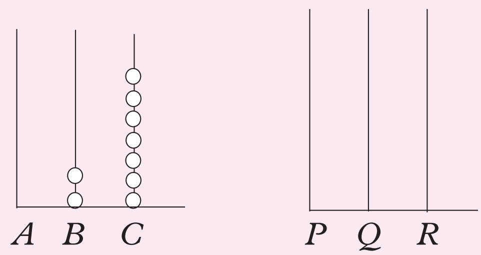


Fig. 2.15

Alternatively,

$$52_8 \rightarrow 50 + 2 \rightarrow 40 + 10$$
$$-\ 23_8 \rightarrow 20 + 3 \rightarrow 20 + 3$$
$$20 + 7 = 27_{eight}$$

Exercise 2.5

1. Work out the following in base eight:
   (a) 17 + 211 (b) 106 + 12
   (c) 257 + 462
2. Evaluate the following in base six:
   (a) 31 – 25 (b) 145 – 51
   (c) 55 – 43 (d) 403 – 54
3. Evaluate the following in base nine:
   (a) 122 + 85 (b) 103 – 86
   (c) 17 – 8 (d) 66 + 35

4. The following calculations are correct. State the base used in each case.

(a) $36 + 26 = 64$ (b) $53 + 36 = 111$ (c) $3 + 23 = 31$

(d) $172 - 169 = 3$ (e) $82 - 53 = 28$ (f) $65 + 36 = 26$

5. Each of the following calculations were done using a certain base. Three of them are correct.

Identify:

(a) the base

(b) the incorrect ones and explain why.

(i) $22 - 16 = 6$ (ii) $68 + 15 = 84$

(iii) $100 - 64 = 25$ (iv) $177 + 19 = 207$

2.3.2 Multiplication

Activity 2.8

Table 2.4 shows part of the multiplication table for numerals in a certain base.

x	0	1	2	3	4	5	6	7	8
0									
1					4				
2									17
3		3		10					
4									
5							23		
6				20					
7									
8			17						71

Fig. 2.4

(a) Identify the base.

(b) Copy and complete the table.

(c) Given that x is a numeral, use your table to find the value of x if:

(i) $6x = 13$ (ii) $3x = 23$

(iii) $7x = 46$

(d) Use your table to formulate three equations using a variable of your choice.

Note that for any base;

(i) the highest numeral is always one less than the base and

(ii) the least is zero (0).

Consider the product:

$2_{six} \times 3_{six}$

Whether in base ten or base six, 2 by 3 remains the same

$\therefore 2 \times 3 = 6_{ten} = 10_6$

Example 2.8

Use long multiplication to evaluate $45_{six} \times 23_{six}$.

Solution

$$\begin{array}{r} 45 \\ \times \quad 23 \\ \hline 1340 \\ + \quad 223 \\ \hline 2003 \\ \hline \end{array}$$

(i) 1st row products

$5 \times 2 = 10_{ten} = (14_{six}$, *we write 4 and carry 1)*

$2 \times 4 = 8_{ten}$ *(*12_{six} *plus the 1 we carried)*

$= 12_{six} + 1$

$= 13_{six}$ (we write 13_{six})

$45 \times 2 = 134_{six}$

(ii) 2nd row products

$3 \times 5 = 15_{ten} = 23_{six}$ (We write 3 and carry 2)

$3 \times 4 = 12_{ten} = 20_{six}$ (20 plus 2 we carried)

$\Rightarrow 20_{six} + 2_{six} = 22_{six}$

$\therefore 45 \times 3 = 223_{six}$.

Add the products in the 1st and 2nd rows to get 2003_{six}

2.3.3 Division

Activity 2.9

(a) Given that a, b and c are numerals in base ten such that $ab = c$, express:

(i) a in terms of b and c.

(ii) b in terms of a and c.

(iii) Describe the operation used to obtain the results above.

(b) Given that $2_{six} \times 5_{six} = 14_{six}$, express:

(i) 2_{six} in terms of 5_{six} and 14_{six}.

(ii) 5_{six} in terms of 2_{six} and 14_{six}.

What operation have you used to obtain your results?

(c) Make a multiplication table for base six and use it to confirm your findings in part (b) above.

(d) Use the table in (c) above to create more examples of division.

Now consider the example $23_{six} \div 5_{six}$. To do this, you ask yourself, ***'by what can I multiply 5_{six} to obtain 23_{six}?'*** This question can be answered using the multiplication table.

Example 2.9

Evaluate: $15_{six} \div 2_{six}$

Solution

$$\begin{array}{r} 5 \\ 2 \overline{)15} \\ \underline{14} \\ 1 \end{array}$$

$2 \times 5 = 10_{ten} = 14_{six}$

$\therefore 15_{six} \div 2_{six} = 5 \text{ Rem } 1$

We could also divide by first changing the number to base 10, then change back to base 6.

$15_{six} = (1 \times 6) + 5 = 11$

$2_{six} = 2_{ten}$

$\therefore 15_{six} \div 2_{six} = 11_{ten} \div 2_{ten} = 5 \text{ Rem } 1$

$5_{six} = 5_{ten}$ and $1_{six} = 1_{ten}$

$\therefore 15_{six} \div 2_{six} = 5_{six} \text{ Rem } 1$

But this is a long and an unnecessary process.

Exercise 2.6

1. Copy and complete the multiplication table in base eight and use it to answer the questions below.

×	0	1	2	3	4	5	6	7	10
0	0	0	0	0	0	0	0	0	0
1	0								
2	0				10				
3	0		6			17			
4	0								
5	0				24				
6	0	6		22					60
7	0							52	
10	0								

Table 2.5

(a) $52_8 \div 7_8$ (b) $43_8 \div 5_8$

(c) $34_8 \div 7_8$ (d) $20_8 \div 4_8$

2. Evaluate the following:

(a) $15_{six} \times 11_{six}$

(b) $21_6 \times 12_6$

(c) $5_6 \times 5_6$

(d) $1\,333_6 \div 35_6$

3. (a) $2\,122_4 \div 23_4$

(b) $100\,122_4 \div 203_4$

4. (a) $1\,216_8 \div 3_8$

(b) $1\,032_6 \div 4_6$

2.4 Special bases

2.4.1 The binary system (base two)

Base two

Activity 2.10

1. Write down all the digits used in the base 10 system.
2. Convert each of the digits in (a) to base 5.
3. Present your findings in a table similar to table 2.6.

Base 10					
Base 2					

Table 2.6

A **binary** system is a number system that uses only two digits 0 and 1. Numbers are expressed as powers of 2 instead of powers of 10 as in the decimal system. Computers use binary notation, the two digits corresponding to two switching position, on and off, in the individual electronic devices in the logic circuits.

Remember; in any base there is no numeral equal to the base. Such a numeral always takes the form of 10.

Base ten:	0	1	2	3	4	5	6	7	8	9	10
Base two	0	1	10	11	100	101	110	111	1000	1001	1010

Table 2.7

Example 2.10

Calculate in binary.

(a) 101 + 10101 *(b)* 1001 – 101

(c) 1101 × 11

(d) 1010101 ÷ 101

Solution

(a)
```
   101
+ 10101
  11010
```

(b)
```
  1001
 – 101
   100
```

(c)
```
    1101
 ×    11
    1101
   1101
  100111
```

(d)
```
          10001
  101 | 1010101
        101
          00101
            101
            000
```

Note: *Just as in division in decimal system, remember to put a zero in the answer any time the divisor fails to divide.*

Exercise 2.7

1. Evaluate:

(a) $1011_2 + 1101_2$

(b) $10001_2 + 110011_2$

(c) $11101_2 + 11_2 + 10101_2$

(d) $1_2 + 11_2 + 1011_2 + 110011_2$

2. Calculate:

(a) $10111_2 - 1101_2$

(b) $11000_2 - 1110_2$

(c) $11111_2 - 10010_2$

(d) $1010101_2 - 1111_2$

3. Evaluate:

(a) $101_2 \times 11_2$

(b) $1111_2 \times 1101_2$
(c) $10101_2 \times 11_1$
(d) $1110_2 \times 111_2$

4. Evaluate:
(a) $101011_2 \div 11_2$
(b) $11100101_2 \div 101_2$
(c) $10001011_2 \div 1011_2$
(d) $100010011_2 \div 101_2$
(e) $100100001_2 \div 10_2$

5. Find the prime factors of 1011100_2.

6. Convert the following to the binary system.
(a) 18_{ten} (b) 135_{six}
(c) 65_{seven} (d) 35_{eight}

7. Convert 10110_{two} to base four.

8. Evaluate the following giving your answers in base two.
(a) $15_{ten} + 23_{ten}$ (b) $35_{ten} - 12_{ten}$

2.4.2 Base twelve (Duodecimal system)

Activity 2.11

1. Think of examples of items where we group in twelves.
2. Use your dictionary to find the meaning of the word dozen.

A system of numbers whose base is twelve is called **duodecimal system**. When buying or selling in bulk, often, items are counted in groups of twelve i.e. dozens. Earlier in the chapter, we saw that the numeral equivalent to the base is always represented by 10. Therefore, in the case of base twelve, we have to define two different variables to use in place of 10 and 11 to avoid confusion. Such substitutions are necessary when working with any base greater than 10, i.e. base eleven, thirteen etc. To be able to list the digits used in base twelve, let letter A represent 10, and B represent 11.

Thus, the digits in base twelve are:
0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B.

Example 2.11

Evaluate the following in base 12.
(a) $12A + 4AB$ (b) $23B \times A$
(c) $789 - AB$

Solution

(a)
$$\begin{array}{r} 12A \\ +\ 4AB \\ \hline 619_{12} \end{array}$$

$A + B = 10 + 11 = 21_{ten} = 19_{12}$
Write 9 carry 1.
$1 + 2 + A = 3 + 10 = 13_{ten} = 11_{12}$
Write 1 carry 1: $1 + 4 + 1 = 6_{12}$

(b)
$$\begin{array}{r} 23B \\ \times\ A \\ \hline 1B32 \end{array}$$

$A \times B = 10 \times 11 = 110_{ten} = 92_{12}$
Write 2 carry 9.
$3A + 9 = 39_{ten} = 33_{12}$
Write 3 carry 3: $2A + 3 = 23_{ten} = 1B_{12}$

(c)
$$\begin{array}{r} 589 \\ -\ AB \\ \hline 49A \end{array}$$

$9 - B = 9 - 11$ (not possible)
Borrow 1 ($\equiv 12$) from 8.
$(9+12)-11 = 21-11 = 10_{ten} = A$ in base 12
$7 - A = 7 - 10$ (not possible)
Borrow 1 ($\equiv 12$) from 5
$(7+12)-10 = 19-10 = 7$
Drop down 4

Example 2.12

Convert $1\,789_{ten}$ to base 12.

Solution

12	1 789	
12	149	Rem 1
12	12	Rem 5
12	1	Rem 0
	0	Rem 1

$1\,789_{ten} = 1\,051_{twelve}$

Exercise 2.8

In question 1 to 4, A represents 10 and B represents 11.

1. Express the following in decimal.
(a) 97_{12} (b) AB_{12}
(c) $9A_{12}$ (d) $B7_{12}$

2. Evaluate:

(a) $AB_{12} + 99_{12}$

(b) $1011_{12} + A0B0_{12}$

3. Calculate:

(a) $B9_{12} - A8_{12}$

(b) $419_{12} - AB_{12}$

4. Multiply:

(a) $B1A_{12}$ by $A01_{12}$ (b) $8A_{12} \times 9B_{12}$

5. Convert: $1\ 332_{four}$ to base twelve.

6. Convert $1\ 705_{ten}$ to base twelve.

Given that A stands for 'ten' and B stands for 'eleven', answer the following questions.

7. Evaluate: $7A_{12} + B5_{12}$. Convert your answer to base 10.

8. Perform the following duodecimal calculations.

(a) $5BA - BA$ (b) $A5 + 5A + 9$

(c) $A \times 64$ (d) $B \times 45$

(e) $12 \times 8 - 7$

2.5 Solving equations involving numbers in other bases

Activity 2.12

Solve the equations;

(a) (i) $x - 6 = 10$ (ii) $x - (-3) = -5$

(iii) $\frac{x}{4} + 3 = 5$

given that you are working with base 10 system.

(b) Solve the equations in (a) above using base 6.

Now, consider the equations below.

(i) $x + 5 = 12$ (ii) $x - (-2) = 4$

(iii) $\frac{2x}{3} + 3 = 5$

Working with base ten system,

(i) $x + 5 = 12 \Rightarrow x = 7$

(ii) $x + 2 = 4 \Rightarrow x = 2$

(iii) $\frac{2x}{3} + 3 = 5 \Rightarrow x = 3$

Now, working in base six,

(i) $x + 5 = 12$ can be written as

$x + 5 = 20$ (since $12_{10} = 20_6$

$\Rightarrow x + 5 = 20$

$x = 20 - 5$

$= 11_{six}$

Alternatively, we can assume that $x + 5 = 12$ is already in base six

$x + 5 = 12$

$x = 12 - 5$

$= (6 + 2) - 5$

$x = 3_{six}$

(ii) $x - (-2) = +4$

$x + 2 = 4$

$x = -2 + 4$

$= 2_{six}$

(iii) $\frac{2x}{3} + 3 = 5$.....$3 \times 3 = 9_{10} = 13_{six}$

And $5 \times 3 = 15_{10} = 23_{six}$

Thus $\frac{2x}{3} + 3 = 5$ becomes

$2x + 13 = 23$ (base six)..... multiplying each term by 3

$2x = 23 - 13$.......... subtracting 13_{six} from both sides

$= 10_{six}$

$x = 3_{six}$........... after dividing both sides by 2_(six)

Example 2.13

Solve for the unknown in:

(a) $2x + 15 = 17$ (base 8)

(b) $\frac{1}{2}x - 3x = 20$ (base 6)

(c) $\frac{3}{4}x + \frac{3}{2}x = 9$(base 10)

Solution

(a) $2x + 15 = 27$

$2x = 27 - 15$

$x = 12 \Rightarrow x = 6$

$x = 6_8$

(b) $-\frac{1}{2}x + 3x = 25_6$

$-x + 10x = 54$

$5x = 54_6$

$x = \frac{54}{5} = 10\frac{4}{5}$

(c) $\frac{3}{4}x + \frac{3}{2}x = 9_{10}$

$3x + 6x = 36$

$9x = 36$

$x = 4$

Example 2.14

Solve for x in the equation $36_x + 26_x = 64_x$ given that x is a number other than base ten. Verify your answer.

Solution

$36_x + 26_x = 64_x$ can be expressed as

$3 \times x + 6 + 2 \times x + 6 = 6 \times x + 4$

(convert the equation to base ten then solve for x)

$3x + 6 + 2x + 6 = 6x + 4$

$5x + 12 = 6x + 4$

$12 - 4 = 6x - 5x$

$8 = x$

Thus $x = 8$.

To verify the answer, substitute $8 = x$ in the given equation $36_x + 26_x = 64_x$

In $36_x + 26_x = 64_x$,

LHS $36_x + 26_x = 3x + 6 + 2x + 6$

$= 3 \times 8 + 6 + 2 \times 8 + 6$

$= 24 + 6 + 12 + 6$

$= 30 + 22$

$= 52$

RHS $64_x = 6x + 4$

$= 6 \times 8 + 4$

$= 48 + 4$

$= 52$

Thus LHS = RHS = 52

x Represents base 8 in the given equation.

Exercise 2.9

1. Solve for x in:
   (a) $2102_3 = 72_x$
   (b) $110011_{two} = 23_x$
2. Solve for x if $110101_2 = x_8$
3. Given that A and B represent ten and eleven respectively in a certain base x, solve for x in:
   (a) $A7_x + 5B_x = 198_{10}$
   (b) $BA1_x = 1705_{10}$
4. Find x if $10011_{two} = 23_x$.
5. Given that x is the base, solve the equation:
   (a) $25_x + 13_x = 42_x$
   (b) $32_x + 24_x = 100_x$
   (c) $142_x + 33_x = 215_x$
6. Solve for x in the following:
   (a) $12_x - 6_x = 5_x$
   (b) $31_x - 16_x = 12_x$
   (c) $32_x - 24_x = 6_x$
   (d) $142_x - 53_x = 67_x$
7. Given that A and B represent 10 and 11 respectively in base twelve, solve for x in;
   (a) $12A_{12} + 4AB_{12} = x_{ten}$
   (b) $789_{12} - AB_{12} = x_{nine}$

Unit Summary

- A numeral is a symbol for a number for example number twenty five is represented by the numeral 25.
- A numeral is composed of one or more digits. Thus a digit is a single numeral.
- 2 is a numeral mode of a single digit 365 is a numeral, each of 3, 6, 5 is a digit 365 is made up of three digits.
- In base ten, we use nine digits or numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in base 8, we need eight digits i.e 0, 1, 2, 3, 4 , 5, 6, 7 working with number bases, we never use a numeral equal to the base.
- The value of a digit depends on its position in the numeral thus, in a number such as

  452_{ten} the value of 4 is 4×10^2,

  the value of 5 is 5×10 And the value of 2 is simply 2.

  $452 = 4 \times 10^2 + 55 \times 10 + 2$

  Similarly, in 452_{six}, the value of 4 is 4×62, that of 5 is 5×6 and 2 remains 2.

  $\therefore\ 452_{six} = 4 \times 6^2 + 5 \times 6 + 2,$

  $= 176_{ten}.$
- If a base is greater than ten, the numerals above ten must be represented by a simple variable or symbol. For example, in base 12 we need to invent a symbol to represent 10 and 11. We could use A for 10, B for 11 or any other variable provided we define it i.e we could say let

  $A = 10$

  $B = 11$

  The number equal to the base is always represented by 10.

  In base 12, we use 10 for 12, in base 13 we use 10 for 13 and so on.

Unit 2 Test

1. Add 3554_{six} to 44_{six} giving your answer in the same base.
2. (a) Express 101_{eight} in base 2.

   (b) Calculate $110_{two} \times 1010_{two}$ giving your answer in base two and also in base ten.
3. Write 230_n as an algebraic expression in terms of n.
4. Given that $10022_{three} = 155_n$, find the value of n.
5. Write each of the numbers as a mixed number in the base number

   (a) 101.11_{ten}

   (b) 21.01_{five}
6. If $13 \times 21 = 303$ find the base of the multiplication.
7. In this binary addition, t and r, stands for a particular digit i.e 0 or 1 find t and r complete the addition.

$$\begin{array}{r} 1trrt \\ +\ \ 1trr \\ \hline 1....r \end{array}$$

8. Carry out the following in base six

   (a) 115 + 251 + 251

   (b) 53412 - 34125

   (c) 123×54
9. If A stands for 10 and B stands for eleven, perform the following duodecimal calculations:

   (a) $59A + AB$ (b) $4A + AB + 9$

   (c) 10×54 (d) $45B - A1$

   (e) $11 \times 7 - 8$ (f) $32 + 6B$

   (g) $159A - 6BA$
10. Solve the equation:

    $31_x - 17_x = 16_x$

3 ALGEBRAIC FRACTIONS

Key unit Competence: By the end of the unit, the learner should be able to perform operations on rational expressions and use them in different situations.

Unit Outline

- Definition of algebraic fraction
- Simplification of algebraic fractions
- Subtraction and addition of algebraic fractions with linear denominator
- Multiplication of algebraic fractions
- Division of algebraic fractions
- Solving of rational equations.

Introduction

Unit Focus Activity

(a) Consider a fraction such as $\frac{2}{2x-4}$.

(i) Find the value of the fraction when $x = 0, 1, 2, 3, 4$.

(ii) Is there any value of x for which there cannot be any meaningful value for the fraction in (i) above? If your answer is yes, explain.

(b) Given group of numbers such as (i) 2, 3, 4 (ii) x, $2x$, $2x + 6$, find the LCM of each group.

Express each of the following as a single fraction under a common denominator:

$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$; $\frac{1}{2}-\frac{1}{3}$;

$\frac{1}{x}+\frac{1}{2x}+\frac{2}{2x+6}$; $\frac{1}{x}+\frac{1}{2x}$

(c) (i) Evaluate the following;

$\frac{1}{2}\times\frac{1}{2}$; $\frac{1}{3}\times\frac{1}{3}$; $\frac{3}{4}\times\frac{3}{4}$

(ii) Given that a number divided by itself equals 1, evaluate:

$\frac{1}{2}\div\frac{1}{2}$; $\frac{1}{3}\div\frac{1}{3}$; $\frac{2}{5}\div\frac{2}{5}$; $\frac{a}{b}\div\frac{a}{b}$

(iii) Now evaluate also;

$\frac{1}{2}\times\frac{2}{1}$; $\frac{1}{3}\times\frac{3}{1}$; $\frac{2}{5}\times\frac{5}{2}$; $\frac{a}{b}\times\frac{b}{a}$

What can you say about the answers in part c(ii) and (iii)?

(iv) Create a multiplication question which gives the same answer as each of the following:

$\frac{1}{4}\div\frac{1}{4}$; $\frac{1}{5}\div\frac{1}{5}$; $\frac{2}{3}\div\frac{2}{3}$.

(d) Work out the following:

(i) $a\div\frac{b}{c}$ (ii) $\frac{b}{c}\div a$

(iii) $\frac{a}{b}\div\frac{c}{d}$

Consider the following expressions:

$\frac{x}{4}+\frac{3y}{x}+\frac{m-5}{3}$; $\frac{3(p-1)}{q+1}$; $\frac{5}{x+4}$

In each of these expressions, the numerator or the denominator or both contain a variable or variables. These are examples of algebraic fractions.

Since the letter used in these fractions stand for real numbers, we deal with algebraic fractions in the same way as we do with fractions in arithmetic.

3.1 Definition of algebraic fraction

Activity 3.1

Consider the fractions: $\frac{3y}{1-x}$, $\frac{5}{y+4}$, $\frac{2}{7}$, $\frac{5x-6}{x+3}$, $\frac{x+y}{2}$, $\frac{3}{4}$.

1. Identify the algebraic fractions.
2. Find the value of the variable

that makes each of the following expressions zero:

(i) $x + 3$ (ii) $y + 4$

(iii) $1 - x$

3. What do your answer in (2) above reveal to you about the fractions such as:

$\frac{5x - 6}{x + 3}$, $\frac{5}{y + 4}$ and $\frac{3y}{1 - x}$?

Now consider the following fractions:

(i) $\frac{2}{x}$ (ii) $\frac{x + 3}{x - 1}$ (iii) $\frac{y - 4}{2y - 6}$

The expressions $\frac{2}{x}$, $\frac{x + 3}{x - 1}$ and $\frac{y - 4}{2y - 6}$ are all algebraic fractions.

(i) $\frac{2}{x}$, the fraction is valid for all real numbers **except** when $x = 0$

(ii) $\frac{x + 3}{x - 1}$ exists **only** if $x - 1 \neq 0$

$x - 1 \neq 0$ if $x \neq 1$.

$\therefore$ the fraction is not defined when $x = 1$.

(iii) $\frac{y - 4}{2y - 6}$, the fraction is defined (exists) **only if the denominator is not equal to zero**.

Thus if $2y - 6 = 0$, then $2y = 6$,

$$y = \frac{6}{2} = 3.$$

$\therefore$ $\frac{y - 4}{2y - 6}$ exists for all real values of y except when $y = 3$.

Note:

(a) If $x = 0$ (in (i) above), it means dividing 2 by zero which is not defined.

(b) If $x = 1$ (in (ii) above), the denominator or divisor becomes zero which is not defined/which does not exist.

(c) Similarly, in (iii) if $y = 3$, then $2y - 6 = 0$ (the divisor)which is not defined/ which does not exist.

In general, **an algebraic fraction exists only if the denominator is not equal to zero**. The values of the variable that make the denominator zero is called a **restriction on the variable(s)**. An algebraic fraction can have more than one restriction.

Example 3.1

Identify the restriction on the variable in the fraction $\frac{3xy}{(x + 3)(x - 2)}$.

Solution

In $\frac{3xy}{(x + 3)(x - 2)}$, *there are two factors in the denominator i.e.* $(x + 3)$ *and* $(x - 2)$.

The denominator equals zero if;

$x + 3 = 0$ *or* $x - 2 = 0$

Therefore, $x = -3$ *or* $x = 2$

If $x = -3$, $(x + 3)(x - 2) = 0$ *and*

if $x = 2$, $(x + 3)(x - 2) = 0$

$\therefore$ *In* $\frac{3xy}{(x + 3)(x - 2)}$ *the restrictions are* $x \neq -3$ *or* $x \neq 2$.

Exercise 3.1

1. Identify the restrictions on the variables of each of the following fraction.

(a) $\frac{a + 2}{a}$ (b) $\frac{2x}{y - 2}$

(c) $\frac{7}{3m + 6}$ (d) $\frac{3w - 9}{w + 3}$

(e) $\frac{2a + 6}{a + 3}$ (f) $\frac{3a + 6}{1 - 2a}$

(g) $\frac{2x}{x - 2}$ (h) $\frac{2w + 3}{w - 1}$

(i) $\frac{a^2 - 4}{a - 2}$ (j) $\frac{(a^2 - 4)(a + 3)}{a + 3}$

2. Find the restrictions on the variables in:-

(a) $\frac{y^2 - 16}{(y - 4)(y + 8)}$ (b) $\frac{y^2 - 9}{(y - 3)^2}$

(c) $\frac{y^2 - 5y + 6}{(y-3)(y+2)}$ (d) $\frac{10ab - 15a^2b}{4a(3a-2)}$

3. Find the restrictions on the variables in the following fractions:

(a) $\frac{x^2 - 2x - 15}{4(x-5)}$

(b) $\frac{2y-2}{(y-2)(y-1)}$

(c) $\frac{2y-7}{(2y-7)(y+3)}$

(d) $\frac{6x^2 - x - 1}{(2x-1)(x+1)}$

3.2 Simplification of Fractions

Activity 3.2

Write the following fractions in the simplest form:

$\frac{8}{14}, \frac{35}{45}, \frac{12a}{15a^2}, \frac{8a^2b}{16a^3c^2}, \frac{15x^3y^4}{3xy^5}, \frac{8}{12}, \frac{30}{45}$

$\frac{9}{12}, \frac{3ab}{4a^2b}$ and $\frac{12x^2y^4}{3xy^2}$

A fraction is in its simplest form if its numerator and denominator do not have common factors. To simplify means to divide both numerator and denominator by the common factor or factors.

A numerator and a denominator can be divided by the same factor without altering the value of the fraction.

For example, in $\frac{8}{12}$, the numerator and denominator

have a **common factor** 4.

$\therefore \quad \frac{8}{12} = \frac{8 \div 4}{12 \div 4}$

$= \frac{2}{3}$

We say that $\frac{8}{12}$ and $\frac{2}{3}$ are **equivalent fractions**.

Similarly, $\frac{30}{45} = \frac{30 \div 15}{45 \div 15} = \frac{2}{3}$

$\therefore \frac{30}{45}$ is equivalent to $\frac{2}{3}$.

Also, $\frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4}$.

$\therefore \frac{9}{12}$ is equivalent to $\frac{3}{4}$.

In $\frac{3ab}{4a^2b}$, a and b are common factors.

$\therefore \frac{3ab}{4a^2b} = \frac{3ab \div ab}{4a^2b \div ab} = \frac{3}{4a}$

In $\frac{12x^2y^4}{3xy^2}$, 3, x, y^2 are common factors

Therefore, $\frac{12x^2y^4}{3xy^2} = \frac{12x^2y^4 \div 3xy^2}{3xy^2 \div 3xy^2}$

$= \frac{4xy^2}{1} = 4xy^2$.

If both the numerator and denominator of a fraction have more than one term, we simplify the fraction by:

(i) Factorising both numerator and denominator where necessary.

(ii) Cancelling by the common factor.

Remember: In Senior 2, you learnt to factorise algebraic expressions.

Examples 3.2

Simplify the following fractions:

(a) $\frac{8x^2y^3}{2x^3y}$ (b) $\frac{2x^2 + 5x^3}{2x^2 + 4x^3}$

Solution

(a) $\frac{8x^2y^3}{2x^3y}$

$8x^2y^3$ is the numerator and

$2x^3y$ is the denominator

The common factors are 2, x^2y

$\therefore \frac{8x^2y^3}{2x^3y} = \frac{8x^2y^3 \div 2x^2y}{2x^3y \div 2x^2y}$

$= \frac{4y^2}{x}$

(b) $\frac{2x^2 + 5x^3}{2x^2 + 4x^3}$

Both the numerator and the denominator contain two terms each.

$2x^2 + 5x^3 = x^2(2 + 5x)$

$2x^2 + 4x^3 = 2x^2(1 + 2x)$

$$\frac{2x^2 + 5x^3}{2x^2 + 4x^3} = \frac{x^2(2 + 5x)}{2x^2(1 + 2x)}$$ *(divide both numerator and denominator by x^2)*

$$\frac{2 + 5x}{2(1 + 2x)} = \frac{1}{2}\,\frac{(2 + 5x)}{1 + 2x}$$

The Activity 3.3 introduces fractions involving quantratic terms which can be expressed as products of linear expressions.

Activity 3.3

Factorise the following expressions:-

(a) $x^2 - 81$ (b) $3x^2 - 3$

(c) $x^2 - x - 12$ (d) $2x^2 - 9x + 10$

Now consider the following expressions:

(a) $x^2 - 144$ (b) $2x^2 - 2$

(c) $x^2 - 11x + 28$ (d) $2x^2 + 11x + 12$

Factorise completely:

(a) $x^2 - 144$ is a difference of two squares.

$\therefore x^2 - 144 = x^2 - 12^2$

$= (x - 12)(x + 12)$ (factors of a difference of two squares)

(b) $2x^2 - 2 = 2(x^2 - 1)$ (2 is a common factor)

$\therefore 2x^2 - 2 = 2(x - 1)(x + 1)$ is a difference of two squares.

(c) $x^2 - 11x + 28$ is a quadratic expression.

$x^2 - 11x + 28 = x^2 - 7x - 4x + 28$ (split the middle term)

$x^2 - 11x + 28 = x(x - 7) - 4(x - 7)$ (factorise by grouping)

$x^2 - 11x + 28 = (x - 7)(x - 4)$

(d) $2x^2 + 11x + 12$

$= 2x^2 + 8x + 3x + 12$ (Split middle term)

$= 2x(x + 4) + 3(x + 4)$ (factorise by grouping)

$= (x + 4)(2x + 3)$

Examples 3.3

Simplify $\frac{2x - 2}{x^2 - 3x + 2}$ *and note the restrictions.*

Solution

The numerator has two terms 2x and –2.

So, $2x - 2 = 2(x - 1)$ (factor out the common factor 2).

The denominator is a quadratic expression

$x^2 - 3x + 2 = x^2 - 2x - x + 2$ *(splitting the middle term)*

$= x(x - 2) - 1(x - 2)$ *(factorise by grouping)*

$= (x - 2)(x - 1)$

Restriction are $x \neq 2$ and $x \neq 1$

$$\therefore \frac{2x - 2}{x^2 - 3x + 2} = \frac{2(x - 1)}{(x - 2)(x - 1)}$$

$(x - 1)$ is a common factor in both numerator and denominator.

$$= \frac{2(x - 1)}{(x - 2)(x - 1)}$$ *cancel out the common factors*

$$= \frac{2}{x - 2}$$

Exercise 3.2

Simplify the following fractions:

1. $\frac{3x^3}{2x^2y}$ **2.** $\frac{pqr^2}{p^2qr}$ **3.** $\frac{8x^2}{12x^3}$

4. $\frac{3x^2}{4x^4}$ 5. $\frac{2x^2}{3xy}$ 6. $\frac{abc}{bcd}$

7. $\frac{-abc}{-bcd}$

For each of the following fractions in question 8 and 9 below;

(i) write the expression in factor form

(ii) note the restrictions on the variables

(iii) simplify the fractions.

8. (a) $\frac{x^2 - 2x - 15}{4x - 20}$

(b) $\frac{2y - 7}{2y^2 - y - 21}$

(c) $\frac{6x^2 - x - 1}{4x^2 - 1}$

(d) $\frac{x^2 - 5x + 6}{x - 3}$

(e) $\frac{y^2 - 7y + 6}{y - 6}$

(f) $\frac{x^2 - 9x + 20}{x - 5}$

9. (a) $\frac{x^2 - 10xy + 25y^2}{x + 5y}$

(b) $\frac{6x^2 - 5xy + y^2}{6x^2 - xy - y^2}$

(c) $\frac{x^2 - y^2}{3x^2 - 2xy - y^2}$

(d) $\frac{2x^2 + xy - y^2}{4x^2 - y^2}$

3.3 Addition and subtraction of algebraic fraction with linear denominator

Activity 3.4

1. Find the LCM of 2, 6 and 7.
2. Express the fractions $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{7}$ as equivalent fractions i.e. with a common denominator. Hence find the value of $\frac{1}{2} + \frac{1}{6} + \frac{1}{7}$
3. Find the value of $\frac{1}{2a} + \frac{1}{6b} + \frac{1}{7c}$.
4. Write your answer in the simplest form.

Remember: To add or to subtract simple fractions, first, find the LCM of the denominator then convert each fraction into a fraction having this LCM as denominator. Add or subtract the numerators and simplify your answer.

Now let us consider numbers $3a$, $4b$, $5c$, $\frac{2}{3a}$, $\frac{1}{4b}$, $\frac{1}{5c}$; $\frac{2}{3a} + \frac{1}{4b} + \frac{1}{5c}$.

To find the LCM of two or more numbers, first express each number as a product of its prime factors.

$3a = 3 \times 1 \times a$

$4b = 2 \times 2 \times b = 2^2b$

$5c = 5 \times 1 \times c$

LCM of $3a$, $4b$ and $5c$ is $3a \times 2^2b \times 5c = 60\,abc$

Finding equivalent factions means expressing with common denominator.

$$\frac{2}{3a} = \frac{2 \times 20bc}{3a \times 20bc} = \frac{40bc}{60abc}$$

$$\frac{1}{4b} = \frac{1 \times 15ac}{4b \times 15ac} = \frac{15ac}{60cab}$$

$$\frac{1}{5c} = \frac{1 \times 12ab}{5c \times 12ab} = \frac{12ab}{60cab}$$

$$\therefore \frac{2}{3a} + \frac{1}{4b} + \frac{1}{5c} = \frac{40bc}{60abc} + \frac{15ac}{60bac} + \frac{12ab}{60\,abc}$$

$$= \frac{40bc + 15x + 12ab}{60abc}$$

Note:

1. Addition of algebraic fractions is performed in the same way in the activity 3.4 above.
2. You can only add fractions if their denominators are the same.
3. The basic rule governing fractions is that numerator and denominator

can be multiplied by the same factor without altering the value of the fraction.

Your skills in arithmetic should extend to skills in algebra.

Now consider $\frac{1}{y} + \frac{1}{x}$.

What is the LCM of x and y.

The LCM of x and y is xy.

$\frac{1}{x} + \frac{1}{y} = \frac{1 \times y}{x \times y} + \frac{1 \times x}{y \times x}$

(Multiply numerators together and denominators together.)

$= \frac{y}{xy} + \frac{x}{yx}$

$= \frac{y + x}{xy}$

Example 3.4

Simplify: $\frac{5}{4ab} + \frac{-2}{3a}$

Solution

Restrictions in those fractions are $a \neq 0$ and $b \neq 0$.

$\frac{5}{4ab} + \frac{-2}{3a} = \frac{5}{4ab} - \frac{2}{3a}$

LCM of 4ab and 3a is 12ab.

$\frac{5}{4ab} - \frac{2}{3a} = \frac{5 \times 3}{4ab \times 3} - \frac{2 \times 4b}{3a \times 4b}$

$= \frac{15}{12ab} - \frac{8b}{12ab}$

$= \frac{15 - 8b}{12ab}$ *(Subtract the numerators.)*

This fraction cannot be simplified further.

Example 3.5

Simplify $\frac{x - 1}{3} - \frac{x - 2}{4}$.

Solution

The denominators are 3 and 4.

The LCM of 3 and 4 is 12.

Now express the fractions with 12 as the denominators.

$\frac{x - 1}{3} - \frac{x - 2}{4} = \frac{4 \times (x - 1)}{4 \times 3} - \frac{3 \times (x - 2)}{3 \times 4}$

$= \frac{4(x - 1) - 3(x - 2)}{12}$

$= \frac{4x - 4 - 3x + 6}{12}$

$= \frac{4x - 3x - 4 + 6}{12}$

$= \frac{x + 2}{12}$

Example 3.6

Work out: $\frac{1}{x + 1} - \frac{1}{2x + 2}$

Solution

The denominators are $(x + 1)$ and $(2x + 2)$.

To find the LCM, we factorise the denominators.

Restrictions in these fractions are

$x \neq -1$

$(x + 1)$ is a prime expression.

$(2x + 2)$ has two terms with common factors 2.

$\therefore\ 2x + 2 = 2(x + 1)$

The factors in the denominator are 2, and $(x + 1)$

$\therefore$ *LCM* $= 2(x + 1)$

$\therefore\ \frac{1}{x + 1} - \frac{1}{2x + 2} = \frac{1}{x + 1} - \frac{1}{2(x + 1)}$

$= \frac{2 \times 1}{2(x + 1)} - \frac{1 \times 1}{2(x + 1)}$

$= \frac{2}{2(x + 1)} - \frac{1}{2(x + 1)}$

$= \frac{2 - 1}{2(x + 1)}$ *adding the numerators*

$= \frac{1}{2(x + 1)}$

Exercise 3.3

1. Find the least common multiple of each of the following:-
   (a) $4x,\ 6x$ (b) $3m,\ 2n$
   (c) $xy,\ x$ (d) $a, b, 2ab$
   (e) xy, x, x^2y (f) $2b,\ 4b.$

Simplify the fractions in the following questions.

2. (a) $\frac{3x}{2} + \frac{5x}{4}$ (b) $\frac{3}{2a} + \frac{5}{4a}$
   (c) $\frac{3x}{2a} + \frac{5x}{3a}$ (d) $\frac{2x}{m} - \frac{3x}{n}$
   (e) $\frac{6}{x} + \frac{2}{xy}$ (f) $\frac{8}{x^2y} + \frac{6}{xy^2}$
3. (a) $\frac{3}{x} - \frac{5}{2x} + \frac{3}{x}$
   (b) $\frac{3a}{x} - \frac{5a}{2x} + \frac{3a}{2}$
   (c) $\frac{x+1}{x} - \frac{3}{4}$
   (d) $\frac{x-1}{3} - \frac{3}{2}$
4. (a) $\frac{2}{3x} - \frac{4}{x} + \frac{5}{6x}$
   (b) $\frac{a}{3m} - \frac{b}{2m} + \frac{2c}{6n}$
   (c) $\frac{5}{2a} - \frac{4}{3a} - \frac{5}{6a}$
5. (a) $\frac{6x+1}{3} - \frac{2x-3}{4}$
   (b) $\frac{8x-2}{9} - \frac{x-3}{6}$
   (c) $\frac{2x-1}{6} - \frac{3x+1}{4} - \frac{2x-5}{2}$
6. (a) $\frac{3}{x-2} - \frac{x}{x-1}$
   (b) $\frac{3}{x-3} - \frac{8}{x+3}$
   (c) $\frac{3}{x-1} - \frac{x}{1-x}$
   (d) $\frac{1}{b} - \frac{1}{b-c}$
   (e) $\frac{1}{2} - \frac{b}{a-2b}$
   (f) $\frac{1}{a+2} - \frac{1}{a-2}$
   (g) $b + \frac{bc}{b-c}$
   (h) $y + z + \frac{z^2}{y-z}$
7. (a) $\frac{3}{a-b} - \frac{4}{a+b}$
   (b) $\frac{2}{y-1} - \frac{3y}{y+5}$
   (c) $\frac{1}{x+3} - \frac{1}{x-3}$
   (d) $\frac{3}{x+1} - \frac{2}{x-1}$
   (e) $1 - \frac{x}{x-y}$
   (f) $\frac{c}{c-d} - \frac{c}{c+d}$
   (g) $r - \frac{r^2}{r+s}$
8. (a) $\frac{1}{y} + \frac{1}{z} - \frac{1}{y+z}$
   (b) $\frac{b}{b+c} - \frac{c}{b-c}$
   (c) $\frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1}$
9. (a) $\frac{r^2-9}{r+3}$ (b) $\frac{y^2-z^2}{z-y}$
   (c) $\frac{y^2-6y+5}{1-y}$

3.4 Multiplication of algebraic fractions

Activity 3.5

(a) Factorise the following expressions.

(i) $2x^3 + 8x^2$ (ii) $x^2 - 16$

(iii) $x^2 + 5x + 6$ (iv) $3x^2 + 6x$

(b) (i) Express $\frac{2x^2+8x^2}{x^2-16}$ in factor form.

(ii) Simplify $\frac{2x^2+8x^2}{x^2-16}$ if possible.

(c) Express $\frac{x^2+5x+6}{3x^2+6x}$ in factor form and simplify if possible.

(d) Express $\frac{2x^3+8x^2}{x^2-16} \times \frac{x^2+5x+6}{3x^2+6x}$ in the simplest form.

In order to multiply fractions, we identify common factors, or possible factors of given expression and divide both the numerator and denominator by the common factors.

For example, to simply an expression such as: $\frac{a^2-4a}{a^2+5a} \times \frac{a^2+7a+10}{3a-12}$, we identify the two fractions $\frac{a^2-4a}{a^2+5a}$ and $\frac{a^2+7a+10}{3a-12}$ have no common factor is evident.

But it is possible to factorise the expressions in both numerators and denominators. In Senior 2 we learned how to factorise.

(a) Now factorise the following:

(i) $a^2 - 4a$ (ii) $a^2 + 5a$

(iii) $3a - 12$

(iv) $a^2 + 7a + 10$

(b) Express $\frac{a^2-z^2}{a^2-5a} \times \frac{x^2-5x-6}{3x^2-6x}$ in factor form.

Thus:

$a^2 - 4a$ is a quadratic expression with only 2 terms, a is a common factor.

$\therefore\ a^2 - 4a = a(a-4)$

$a^2 + 5a$ is another quadratic expression with two terms whose common factor is a.

$\therefore\ a^2 + 5a = a(a+5)$

$3a - 12$ is a linear expression 3 is a common factor.

$\therefore 3a - 12 = 3(a-4)$

$a^2 + 7a + 10$ is a quadratic expression with three terms.

$a^2 + 7a + 10 = (a+2)(a+5)$

Thus, $\frac{a^2-4a}{a^2+5a} \times \frac{a^2+7a+10}{3a-12}$ can be expressed as $\frac{a(a-4)}{a(a+5)} \times \frac{(a+5)(a+2)}{3(a-4)}$.

Note that there are some common factors in both numerator and denominator which can cancel out.

$$\frac{a^2-4a}{(a^2+5a)} \times \frac{(a^2+7a+10)}{3a-12}$$

$$\frac{(a+2)}{3}$$

Example 3.7

Multiply: $\frac{y^2+y}{y^2-4y} \times \frac{y^2-4y-21}{y+3}$

Solution

Restrictions on the variable are $y \neq 4$, or -3 or 0

There are no visible factors, but we can factorise the expression in both numerator and denominator.

$y^2 + y$ is a binomial with a common factor y

So $y^2 + y = y(y+1)$ factor out common factor y

$y^2 - 4y = y(y-4)$ factor out common factor y

$y^2 - 4y - 21$ is a quadratic expression which can be expressed in the form $y^2 - 7y + 3y - 21$ …… (split the middle term,

$y(y-7) + 3(y-7)$ …….. factorise by grouping

$$= (y-7)(y+3).$$

$y + 3$ is a prime expression.

$$\therefore \frac{y^2-y}{y^2-4y} \times \frac{y^2+4y-21}{y+3}$$

$$= \frac{y(y+1)}{y(y-4)} \times \frac{(y-7)(y+3)}{y+3}$$

$$= \frac{(y+1)(y-7)}{y-4}$$

Exercise 3.4

1. Simplify:
   (a) $\frac{36}{y^2 + 2y} \times \frac{y + 2}{9}$
   (b) $\frac{a^2 - b^2}{a^2 - 16} \times \frac{a - 4}{a - b}$
   (c) $\frac{3x - 12}{4x + 20} \times \frac{x^2 + 5x}{x^2 - 4x}$
   (d) $\frac{x^2 - x}{2x + 2} \times \frac{6x}{x^2 - 1}$
   (e) $\frac{x^2 - y^2}{x^2 - 2xy - y^2} \times \frac{1}{xy + y^2}$
   (f) $\frac{y - 3}{2 - y} \times \frac{y - 2}{3 - y}$
   (g) $\frac{x^2 y}{2x - 2} \times \frac{2}{xy^2}$
   (h) $\frac{x + 1}{x^2 - 1} \times \frac{x - 1}{x}$
   (i) $\frac{x - y}{16} \times \frac{8}{x^2 - y^2}$
2. Workout each of the following expressions and simplify the result.
   (a) $\frac{m}{n} \times \frac{b}{a}$
   (b) $\frac{-6}{5} \times \frac{-x}{3}$
   (c) $-3m^2 \times \frac{-5}{6m}$
   (d) $\frac{-6x}{3} \times \frac{-3}{-6x}$
   (e) $\frac{3a^2}{5b^2} \times \frac{10b}{4b^3}$
   (f) $\frac{6}{x^2 y} \times \frac{x^3 y^2}{x}$
   (g) $\frac{x}{3m} \times \frac{-3m}{-y}$
   (h) $\frac{3x^2}{2y} \times \frac{-2y}{3}$
   (i) $\frac{-x^2}{6y} \times \frac{12xy}{-x} \times \frac{2y}{-x^2}$
   (j) $\frac{12x}{-5x^2} \times \frac{-x^2}{6x^2} \times \frac{-10x}{3x}$
3. Multiply and simplify:
   (a) $\frac{x^2 + 3x + 2}{3} \times \frac{9}{x^2 - x - 6}$
   (b) $\frac{x^3 + 4x^2}{x^2 - 1} \times \frac{x^2 - 5x + 6}{x^2 - 3x}$

3.5 Division of algebraic fractions

Activity 3.6

1. List 5 pairs of numbers whose product is 1.
2. Consider the following pairs of numbers:
   $3, \frac{1}{3}$; $2, \frac{1}{2}$; $\frac{5}{2}, \frac{2}{5}$; $a, \frac{1}{a}$; $\frac{a}{b}, \frac{b}{a}$
   (a) Evaluate the product of each pair.
   (b) What do you notice?

In general,
if a and b are whole numbers, then
$\frac{1}{a}$ is the reciprocal of a and
$\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$.
In order to divide a fraction, we must be able to identify the reciprocal of the divisor.

Example 3.8

Find the reciprocal of:
(a) 5 *(b)* $\frac{3}{4}$ *(c)* $\frac{5x}{6y}$

Solution

In this example, we are looking for a number which when multiplied by the given number gives 1.

(a) *Let the reciprocal of 5 be x.*
This means $5 \times x = 1$
$\therefore x = \frac{1}{5}$ *dividing both sides by 5.*
$\therefore$ *the reciprocal of 5 is* $\frac{1}{5}$.

(b) *Let the reciprocal of* $\frac{3}{4}$ *be x.*
$\therefore \frac{3}{4} \times x = 1$
$\frac{3x}{4} = 1$
$3x = 4$ *Multiplying both sides by 4.*
$x = \frac{4}{3}$ *Dividing both sides by 3.*

Therefore, the reciprocal of $\frac{3}{4}$ *is* $\frac{4}{3}$.

(c) *Let the reciprocal of* $\frac{5x}{6y}$ *be R.*

This means $\frac{5x}{6y} \times R = 1.$

$(5x) \times R = 1 \times 6y$ *(multiply both sides by 6y)*

$R = \frac{6y}{5x}$ *(divide both sides by 5x)*

Therefore, the reciprocal of $\frac{5x}{6y}$ *is* $\frac{6y}{5x}$.

Example 3.9

Simplify: $\frac{3x^2y}{4a^2} \div \frac{9xy}{5a}$

Solution

In $\frac{3x^2y}{4a^2} \div \frac{9xy}{5a}$, $\frac{9xy}{5a}$ *is the divisor.*

The reciprocal of $\frac{9xy}{5a} = \frac{1 \times 5a}{9xy}$

$= \frac{5a}{9xy}.$

$\therefore \frac{3x^2y}{4a^2} \div \frac{9xy}{5a} = \frac{3x^2y}{4a^2} \times \frac{5a}{9xy}$

(Cancel by x and y.)

$= \frac{3x \times 5}{4a \times 9}$ *(Cancel by 3.)*

$= \frac{5x}{12a}$

Example 3.10

Solve: $\frac{x^2 - 4y^2}{2x^2 - 5xy + 2y^2} \div \frac{y^2 - xy}{2x - y}$

Solution

Start by fractionising the individual expression in the given fractions.

$x^2 - 4y^2$ *is a difference of two squares*

$\therefore\ x^2 - 4y^2 = (x - 2y)\ (x + 2y)$

$2x^2 - 5xy^2 + 2y^2$ *is a quadratic homogeneous expression.*

$\therefore\ 2x^2 - 5xy + 2y^2$

$= 2x^2 - 4xy - xy - 2y^2$

$= 2x\ (x - 2y) - y(\ x - 2y)$

factorise by grouping

$= (x - 2y)\ \ (2x - y)$

$y^2 - xy$ *is a binomial,*

$y^2 - xy = y(y - x)$

$\therefore\ \frac{x^2 - 4y^2}{2x^2 - 5xy - 2y^2} \div \frac{y^2 - xy}{2x - y}$

$= \frac{(x - 2y)\ (x + 2y)}{(x - 2y)\ (2x - y)} \div \frac{y(y - x)}{2x - y}$

$= \frac{(x - 2y)\ (x + 2y)}{(x - 2y)\ (2x - y)} \times \frac{2x - y}{y(y - x)}$

(Cancel common factors.)

$= \frac{(x + 2y)}{y(y - x)}$

Exercise 3.5

1. Write down the reciprocal of each of the following:

(a) $\frac{3y}{x}$ (b) $\frac{2x^2}{y}$ (c) $-2x$

(d) $\frac{2y}{3}$ (e) $\frac{5y}{4}$ (f) $\frac{-3x}{5y}$

Simplify each of the following.

2. (a) $\frac{x}{y} \div \frac{a}{b}$ (b) $\frac{-x}{y} \div \frac{a}{b}$

(c) $\frac{-2}{7} \div \frac{3}{-x}$ (d) $-6 \div \frac{3}{y}$

3. (a) $\frac{x}{-4} \div \frac{-5}{16y}$ (b) $\frac{x}{ab} \div \frac{-x}{ab}$

(c) $\frac{x^2}{3} \div \frac{20x^2y}{-9}$ (d) $-9 \div \frac{21}{ab}$

4. (a) $\frac{x^2 - 5x + 6}{x} \div \frac{x - 2}{x^2 - 3x}$

(b) $\frac{b - c}{c + b} \div \frac{c - b}{b + c}$

(c) $\frac{a^2b + 2ab^2}{ab} \div (a^2 - 4b^2)$

5. (a) $\left(1 - \frac{c}{d}\right) \div \left(1 - \frac{d}{c}\right)$

(b) $\left(x + \frac{x}{y}\right) \div \left(x - \frac{x}{y}\right)$

6. (a) $\left(\frac{-5}{2x} + \frac{-2}{3x}\right) \div \left(\frac{-3}{x} + \frac{5}{-4x}\right)$

(b) $\left(\frac{-2}{3x} + \frac{8}{-2y}\right) \div \frac{-3}{x^2}$

(c) $\left(\frac{3}{2a} + \frac{-2}{3b}\right) \div \left(\frac{2}{b} + \frac{3}{a}\right)$

7. (a) $\left(\frac{-5}{2x} + \frac{-2}{3x}\right) \div \left(\frac{-3}{x} + \frac{5}{4x}\right)$

(b) $\left(\frac{3}{2a} + \frac{2}{3b}\right) \div \left(\frac{2}{b} + \frac{3}{a}\right)$

3.6 Solving rational equations

Activity 3.7

Find the LCM of the following.

(a) 12, 16, 24 (b) a, b

(c) $(a-3), 2a^2 - 18$ (d) $a^2, (a+1)$

(e) b, 6, $3b^2$

Now let us repeat activity 3.7 using:

(a) 6, 8, 16 (b) x, 2

(c) $x - 3, x^2 - 9$ (d) $x, (x+1)$

(e) y, 4, $3y$

The LCM of a group of numbers is the least or smallest number divided by the given numbers.

Thus, we start expressing each number or expression as a product of prime factors.

(a) 6, 8, 16: $6 = 2 \times 3$

$8 = 2 \times 2 \times 2 = 2^3$

$16 = 2 \times 2 \times 2 \times 2 = 2^4$

LCM $= 3 \times 2^4 = 48$

(b) x and 2 are prime numbers.

$\therefore$ LCM of x and 2 is $2x$.

(c) $x + 3$ and $x^2 - 9$ are algebraic expressions.

$x + 3$ is prime, but $x^2 - 9$ is a difference of squares.

$\therefore \quad x^2 - 9 = (x-3)(x+3)$

The prime factors involved are $(x+3), (x-3)$.

$\therefore$ the LCM $= (x+3)(x-3)$.

(d) x and $x + 1$ are prime expressions therefore LCM of x and $(x+1)$ is $x(x+1)$.

(e) y, 4, 3y:

y is prime,

$4 = 2 \times 2 = 2^2$

$3y = 3 \times y \quad \therefore$ LCM $= 2^2 \times 3 \times y$

In order to be able to solve rational equations;

(i) Start by finding the LCM of the denominators in each equation.

(ii) Use the LCM to eliminate the denominators by multiplying each fraction or term by the LCM.

(iii) Then solve the resulting equation.

Example 3.11

Solve the equation $\frac{x+3}{2} - \frac{x+4}{3} = \frac{x-2}{4}$.

Solution

There are three fractions whose denominators are 2, 3, 4.

2 and 3 are prime numbers,

$4 = 2 \times 2 = 2^2$.

Therefore, LCM of 2, 3 and 4 is $2^2 \times 3 = 12$.

To eliminate the denominators, we multiply each fraction by 12 and cancel by the common factors.

$$12\left(\frac{x+3}{2}\right) - 12\left(\frac{x+4}{3}\right) = 12\left(\frac{x-2}{4}\right)$$

This step may be done mentally.

$$6(x+3) - 4(x+4) = 3(x-2)$$
$$6x + 18 - 4x - 16 = 3x - 6$$
$$6x - 4x + 18 - 16 = 3x - 6$$
$$2x + 2 = 3x - 6$$
$$3x - 2x = 2 + 6$$
$$x = 8$$

By substituting 8 for x in the given equation, we can verify that LHS = RHS.

Example 3.12

Solve the equation $\frac{x+5}{x-3} = \frac{x+3}{x-4}$.

Solution

Restrictions are $x \neq 3$ *and* $x \neq 4$.

In $\frac{x+5}{x-3} = \frac{x+3}{x-4}$, *multiply the equation by the least common multiple (LCM).*

$LCM = (x-3)\ (x-4)$

$(x-3)(x-4)\frac{x+5}{x-3} = (x-3)\ (x-4)\frac{x+3}{x-4}$

This step may be done mentally.

$(x-4)\ (x+5) = (x-3)\ (x+3)$

$x(x+5) - 4(x+5) = x^2 - 9$ *(expansion of binomials)*

$x^2 + 5x - 4x - 20 = x^2 - 9$

$x^2 - x^2 + 5x - 4x = -9 + 20$

$x = -9 + 20$

(Collect like terms and simplify)

$\therefore \quad x = 11$

Alternative approach

Since the equation $\frac{x+5}{x-3} = \frac{x+3}{x-4}$ *has only two fractions (terms), we can eliminate the denominators as follows*

Step 1: $(x-4)\frac{x+5}{x-3} = \frac{x+3}{x-4}(x-4)$

Multiply both sides by one of the denominators.

$(x-4)$ *or* $x+3$

Step 2: $(x-3)\ (x-4)\frac{x+5}{x-3} = (x+3)$

$(x-3)$ *Multiply both sides by the second (remaining) denominators.*

Step 3: $(x-4)\ (x+5) = (x+3)$

$(x-3)$ *from here proceed as above.*

In short, this is called cross – multiplication method

Exercise 3.6

1. Solve for x and in each case, state the restrictions on the variable.

   (a) $\frac{2}{3x} + \frac{3}{5} = \frac{1}{4x}$

   (b) $\frac{x+18}{4} + x = \frac{x-5}{2} + 2$

   (c) $\frac{3}{x} + 2 = \frac{2x+3}{x-1}$

   (d) $\frac{x+1}{x} - 1 = \frac{x-2}{2-x} + 2$

2. Solve and verify your answers and state the restrictions in each case.

   (a) $\frac{x+8}{x+3} = \frac{x}{x-3}$

   (b) $\frac{3x-1}{x} = \frac{5}{2}$

   (c) $\frac{x-1}{x} = \frac{x+1}{x+3}$

   (d) $\frac{y+1}{y-2} = \frac{y+3}{y-4}$

3. Determine the value of x in the following equations.

   (a) $\frac{x-1}{x} + \frac{1}{5} = \frac{-4}{5x}$

   (b) $\frac{3x}{x-3} = \frac{3x-1}{x+3} - 2$

   (c) $\frac{3a}{a} = \frac{2a+3}{a-1} - 2$

   (d) $\frac{1}{x} = \frac{2}{x+1} + \frac{1}{x-1}$

   (e) $\frac{x-1}{x} + \frac{1}{4} = \frac{5}{3x}$

   (f) $\frac{2x-2}{x+2} = 2 + \frac{5-x}{2+x}$

4. Work out the following.

   (a) $\frac{2x-1}{3x+1} = \frac{6x-1}{9x-3}$

   (b) $\frac{1}{x-2} + \frac{3}{x-3} + \frac{5}{2-x} = 0$

(c) $3x + 5 = \frac{2x^2 + 5}{5x - 3}$

(d) $\frac{x - 1}{x + 2} + \frac{1 + 3}{x - 2} = 2$

5. Find the value of x:

(a) $\frac{x}{2x - 4} - \frac{x}{3x - 6} = 1$

(b) $(x + 4) - \frac{x^2}{x - 4} = 1$

(c) $\frac{x}{x + 2} = \frac{3}{x - 20} - \frac{2}{x - 5}$

Unit Summary

- An algebraic fraction is defined or said to exist only if its denominator is not equal to zero. For example, a fuction such as $\frac{3}{x}$ is valid for all values of x except when $x = 0$. The value of a variable that makes the denominator of a fraction zero is called a **restriction on the variable**.
- Two algebraic fractions are said to be equivalent if both can be reduced or simplified to the same simpleast fraction. For example, $\frac{2}{4}, \frac{5}{10}$ *and* $\frac{20}{40}, \ldots$ are equivalent, and all are reducible to $\frac{1}{2}$.
- To add or subtract algebraic fractions, we must first express them with a common denominator, which represents the LCM of the denominators of the individual fractions.
- To multiply algebraic fractions, we begin by identifying common factors in both numerators and denominators. The factors may not be obvious in such a case, factorise all the algebraic expressions involved if possible, then proceed to cancel and multiply.
- Division by a fraction, algebraic or atherimise, means multiplying the divided by the receiprocal of the divisor. Remember the product of a fraction and its reciprocal equal to 1. For example, reciprocal of $\frac{1}{2}$ is 2, that of a is $\frac{1}{a}$, that of $\frac{a}{b}$ is $\frac{b}{a}$ and 30 on.
- Algebraicequationsinvolvingfractions are also called rational equations. To solve rational equations, we begun by eliminating the denominations by multiplying all the terms by the LCM of the denominators. Then proceed to solve the resulting equation.

Unit 3 Test

Simplify each of the following algebraic fractions by expressing them as single fractions in their lowest terms.

1. (a) $\frac{a}{3} + \frac{a}{4} + \frac{a}{5}$ (b) $\frac{x}{2} + \frac{x}{3} + \frac{x}{4}$

(c) $1 + \frac{x + 2}{3}$ (d) $\frac{x}{3} + \frac{2x - 1}{4}$

2. (a) $\frac{3a}{4} - \frac{a}{5}$ (b) (a) $\frac{x}{2} - \frac{y - 4}{3}$

(c) $\frac{4x}{3y} - \frac{2x}{5y}$ (d) $\frac{x - 3}{4} - \frac{x}{3}$

3. (a) $\frac{8x}{xy} \times \frac{2x}{4x}$

(b) $\frac{5a \times 2b \times 3a}{6ab} \times \frac{1}{6ab}$

(c) $\frac{3a^2b^3}{2cd} \times \frac{c^3d^2}{ab}$ (d) $\frac{6ab}{c} \times \frac{ad}{2b} \times \frac{8cd^2}{4bc}$

4. (a) $\frac{ab^2}{bc^2} \div \frac{a^2}{3bc^3}$ (b) $\frac{6ab}{c} \div \frac{4a^2}{7bd}$

5. (a) $\frac{4}{x} - \frac{5}{2x} + \frac{3}{4x}$

(b) $\frac{m}{x} - \frac{3m}{2x} + \frac{5m}{3x}$

(c) $\frac{x}{2} - \frac{x-1}{3}$ (d) $\frac{2a+3b}{3} - \frac{a-2b}{2}$

6. Solve:

(a) $\frac{m}{3} + \frac{m}{5} = 2$ (b) $\frac{x}{5} - \frac{x}{3} = 2$

(c) $\frac{3x}{2} + 1 = \frac{3}{4} + 2x$ (d) $\frac{x+3}{4} - \frac{x-3}{5} = 2$

7. Simplify the following:

(a) $\frac{24x^2}{45} \times \frac{-9}{15x}$

(b) $\frac{xy}{-5} \times -\frac{10x}{x-4} \times \frac{2}{y}$

(c) $\frac{2x}{y} \times \frac{-6x}{5y} \times \frac{y^2}{-3x}$

(d) $(\frac{4a}{-3y}) \times (\frac{-3}{8ab}) \times \frac{3b}{2}$

8. Work out:

(a) $\frac{10ab - 15a^2b}{12a^2 - 8a}$

(b) $\frac{a^2 - b^2}{a^2 + b^2} \times \frac{a^2 + ab}{b^2 - a^2}$

(c) $\frac{x^2 + 2x - 3}{3x^2 + 9x}$

(d) $\frac{16 - 4d}{2d^2 - 7d - 4}$

9. Add the following expressions and simplify the results:

(a) $\frac{2-3y}{3y} + \frac{2y^2+1}{2y^2}$

(b) $\frac{x-3}{2x} - \frac{x^2-1}{4x^2}$

10. Multiply and simplify the result.

(a) $\frac{x^2 - 4x}{5x + x^2} \times \frac{x^2 + 7x + 10}{3x - 12}$

(b) $\frac{2x^2}{5}\left(\frac{5}{x} - \frac{-3}{2x^2}\right)$

11. Divide the following:

(a) $\frac{-3x^2y}{4a^2} \div \frac{9xy}{-5a}$

(b) $\left(\frac{-5}{2x} - \frac{-2}{3x}\right) \div \frac{-3}{x} + \frac{5}{-4x}$

12. (a) Calculate the value of $\frac{x+1}{x^2-1} \div \frac{3x}{x-1}$.

(b) What restrictions apply?

13. (a) What is the first step in solving the equation $x + 1 - \frac{2x-1}{3} = 1$?

(b) Solve the equation in (a) and verify your answer.

14. Solve the equations:

(a) $\frac{3x+2}{3x+4} = \frac{x-1}{x+1}$

(b) $\frac{x-1}{x+1} = \frac{2}{x+1} - \frac{1}{x+1}$

In each case list the restrictions that apply.

4 SIMULTANEOUS LINEAR EQUATIONS AND INEQUALITIES

Key unit competence: By the end of this unit, the leaner should be able to solve word problems involving simultaneous linear equations and inequalities.

Unit outline

- Simultaneous linear equation
- Inequalities
  - Graphical representation of linear inequalities
  - Forming inequalities from given regions
  - Linear inequalities in two unknowns
  - Graphical solutions of simultaneous linear inequalities
  - Linear inequalities from inequality graphs

Introduction

Unit Focus Activity

1. (a) On the same axes draw the graph of the lines whose equations are:

   $-2x + y = -1$ ....(i)

   $x + 2y = 4$ ....(ii)

   (b) State the coordinates of the point where the two lines intersect. What meaning do you attach to this answer?

2. Using the same equations in 1 (a) above.

   (a) Use equation (i) and express x in terms of y.

   (b) Substitute your answer from (a) in equation (ii) to obtain an equation in y.

   (c) Solve the above equation to obtain the value of y.

   (d) Substitute the y value in equation (i) to obtain the value of x.

3. Consider the situation below:

   A learning institution employs men and women during the school vacation. A day's wage for 3 men and 2 women is 4 000 FRW. For 1 man and 5 women the wage is 3 500 FRW.

   (a) If a man earns x FRW and a woman y FRW per day, write two equations in terms of x and y for the given situation.

   (b) Solve the equations using different methods.

4. Consider the inequalities:

   (i) $y > x + 1$ (ii) $2x + y \leq 5$

   (a) On a Cartesian plane, draw the line $y = x + 1$. Pick a point clearly not on the line and substitute the x and y values in $y = x + 1$ to determine the region of $y > x + 1$

   On the same axis draw the line $y + 2x = 5$

   By substitution identify the region $y + 2x \leq 5$

   (b) Identify the region that satisfies both the inequalities

   $y > x + 1$ and $y + 2x \leq 5$

4.1 Simultaneous linear equation

4.1.1 Solving simultaneous equations graphically

Activity 4.1

Using the equations

(i) $y = 2x + 1$ (ii) $2y = x - 4$

1. Make separate tables of values for each equation for values of x: $-3 \leq x \leq 3$.
2. On the same axes, draw the two lines.
3. Do the lines intersect? If yes state the coordinates of the point where they intersect.
4. Describe the significance or the meaning of the point of intersection of the two lines.

Points to note

1. Before drawing any graph it is usually necessary to;
   i. Make a table of values for at least three pairs of points.
   ii. Plot points from the table.
2. In each of the equations, there are two variables, x and y. x is called the independent variable, marked on the horizontal axis; y is called the dependent variable marked on the vertical axis.
3. Choose an appropriate scale depending on the values on the table.

   If the number is small we use a small scale such as 1cm represent 1 unit on both sides.
   - We can use a graphical method to solve any pair of simultaneous equations.
   - If a pair of equations produces parallel lines when graphed, then the equations have **no solution** since the lines never intersect.
   - Remember when two lines are coincident, then the lines represent a pair of equations which are said to have an **infinite solution set**. This means that for any value of x picked there is a corresponding value of y on the graph.
   - If the lines intersect, there is one **unique solution**.

Example 4.1

Solve graphically the simultaneous equations $2y + 3x = 12$, $3y = 5x - 1$

Solution

Step 1: Make a table of values for each equation. Remember three pairs of points are sufficient for each equation.

Table 4.1 and 4.2 represent the relations $2y + 3x = 12$ and $3y = 5x - 1$ respectively.

x	0	4	2
y	6	0	3

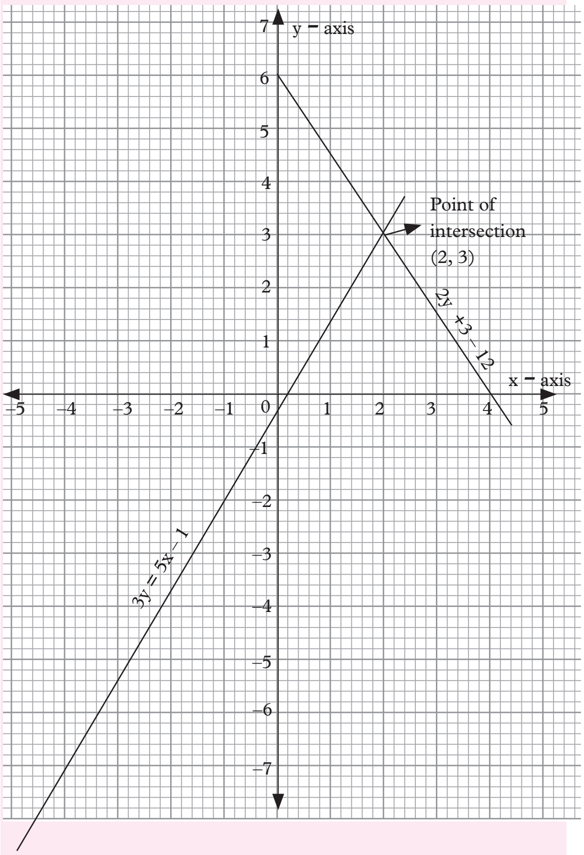
Table 4.1

x	–1	2	–4
y	–2	3	–7

Table 4.2

Step 2: Choose a suitable scale and plot the points on the same axes

- *Draw both lines on the same graph*
- *If the lines do not intersect within the chosen interval extend them until they meet, unless the lines are parallel.*



Graph 4.1

Step 3: Point of intersection has coordinates (2,3) as read from the graph.

The solution of the simultaneous equations $2y + 3x = 12$ and $3y = 5x - 1$ is $x = 2$ and $y = 3$

Now, check that the solutions satisfy the two equations.

Substitute 2 for x and 3 for y in the equations (1) and (2) in turns.

(i) In $2y + 3x = 12$ use 2 for x and 3 for y.

$$LHS = (2 \times 3) + (3 \times 2)$$
$$= 6 + 6$$
$$= 12$$
$$LHS = RHS = 12$$

(ii) In $3y - 5x = -1$ use $x = 2$ and $y = 3$

$$LHS = (3 \times 3) - (5 \times 2)$$
$$= 9 - 10$$
$$= -1$$
$$LHS = RHS = -1$$

We can also solve the equations by another method to verify that we obtain the same solution

Now; solve the two equations using (i) elimination (ii) substitution and comparison methods.

Exercise 4.1

Use graphical method to solve the following simultaneous equations. Verify your answers by substituting your solutions in the given equations.

1. $x + y = 3$
   $4x - 3y = 5$
2. $x - 2y = 5$
   $2x + y = 5$
3. $x + y = 0$
   $2y - 3x = 10$
4. $x + 2y = 5$
   $4x + 2y = 1$
5. $4x - 3y = 1$
   $x - 4 = 2y$
6. $6x - y = -1$
   $4x + 2y = -6$
7. $4x - y = -3$
   $8x + 3y = 4$
8. $5x + 2y = 10$
   $3x + 7y = 29$
9. $2x - 4y = 8$
   $3x - 2y = 8$
10. $\frac{1}{2}x - 2y = 5$
    $\frac{1}{2}x + y = 1$
11. $2x - 4y + 10 = 0$
    $3x + y - 6 = 0$
12. $3x - 5y = 23$
    $x - 4y = 3$
13. Solve questions 1 to 12 using;
    i. Elimination method
    ii. Substitution method
    iii. Comparison method

4.1.2 Solving problems involving simultaneous equations

Activity 4.2

Two numbers x and y are such that $x < y$. The sum of x and y is 90 and a third of the smaller number equals a seventh of the larger.

1. Form a relation connecting x, y and 90.
2. Relate a third of the smaller number and a seventh number of larger in an equation.
3. State two equations in terms of x and y.
4. Solve the equations in (3) above simultaneously.
5. Hence state the value of the two numbers x and y.

In this section we shall deal with situations which give rise to simultaneous equations.

Points to note

- To form simultaneous equations from a given situation, we must define the two variables, say x and y.
- Relate the two variables using the given information i.e. form two distinct equations in x and y.
- Simultaneous equations can be solved either algebraically or graphically.
- To solve simultaneous equations, graphically, we draw the lines representing the two equations on the same graph.
- If the equations have a solution, the lines will intersect at a point. The x and y values at the point of intersection represent the solution of the equations.
- If no solution, the lines will be parallel.
- If the lines are coincident, it means the equations have an infinite solution.

Example 4.2

Some bird watchers travelled along the river for 3 hours and then travelled in the forest for 6 hours. The total distance travelled was 216 km. If they went 12 km/h faster in the forest than along the river, what would have been the different speed?

Solution

Let the speed along the river be x km/h and in the forest y km/hr.

River	Forest
Speed xkm/h	*Speed ykm/h*
Distance 3xkm	*Distance 6y km*

Thus $x + 12 = y$

$3x + 6y = 216$

Equations (1) and (2) represent the relation between the speeds and the distance.

On squared/graph paper represent equations (1) and (2)

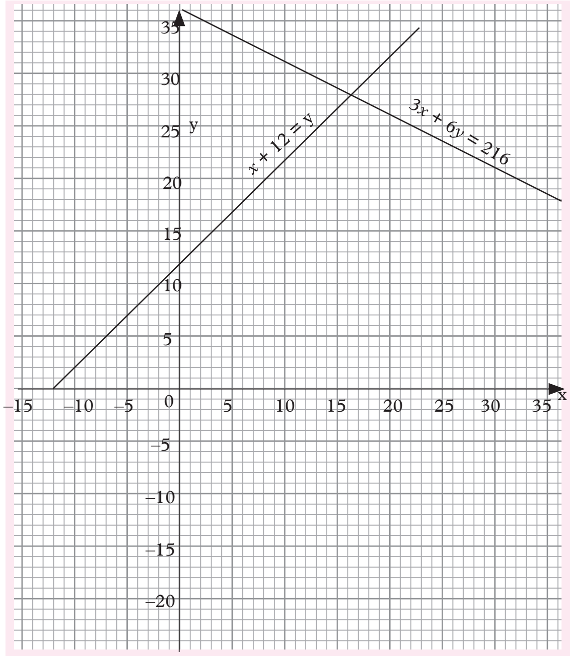
x	*0*	*–12*	*–6*
y	*12*	*0*	*6*

for $x + 12 = y$

x	*0*	*72*	*10*
y	*36*	*0*	*31*

for $3x + 6y = 216$

The graph on the next page shows the required graphical solution of the given problem



Graph 4.2

Exercise 4.2

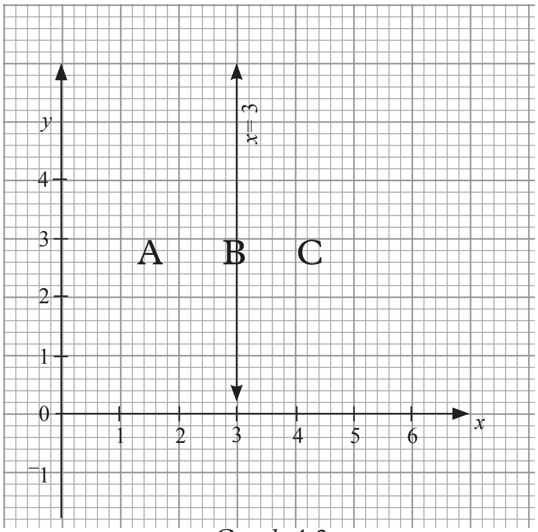
1. Find the total distance in terms of x and y for each of the following:
   (a) A car travels for x hours at 60 km/h and y hours at 100 km/h.
   (b) A person ran for 5 hours at x km/hr and 10 hours at y km/h.
2. Write an equation in two variables for each of the following:
   (a) The total interest on an amount of money invested at 10% p.a and another amount invested at 12% p.a is 1 609 FRW.
   (b) The interest on an amount of money invested at 8% p.a exceeds the interest on another amount of money invested at 9% p.a by 100.
3. Write an equation in two variables for each of the following:
   (a) The sum of two numbers is 48.
   (b) One number exceeds another by 5.
   (c) The sum of the width and length of a rectangle equals 96 units.
   (d) When Jane's age is added to Anne's age, the sum is 36 years.
4. Two numbers are such that their sum is 84 and three times the greater exceeds the twice the smaller by 62. Form two simultaneous equations and solve them to find the numbers.
5. John has a total bill of \$580 consisting of \$5 bills and \$10 bills. If he has a total of 76 bills, how many of each does he have?
6. Mary invested her savings of 4800 FRW partly at 9% p.a and the rest at 10% p.a. At the end of the year the interest from the 9% interest was 4300 FRW less than the interest from the 10% investments. Form a pair of simultaneous equations and solve them to find how much was invested at each rate.
7. James rented a tourist van and went at 40 km/h on all weather road and at 10 km/h through a park. It took 5.75 hours to travel 185 km on the trip. Use simultaneous equations to find the number of kilometres he drove through the park.
8. A newspaper editor hired a writer for jokes or cartoons. The cost for 8 jokes and 6 cartoons is 610 FRW. The cost of 6 jokes and 8 cartoons is 510 FRW. How much do a joke and a cartoon together cost?
9. Two numbers are such that their sum divided by 4 is equal to 14. If the greater number is increased by 24, the result equals three time the smaller number. Find the two numbers.

10. At an environmental studies conference, there were 168 more engineers than chemists. However, there were 268 physicists. If there were a total of 1134 engineers and chemists, how many of each were there?
11. At the beginning of the rainy season, a farmer bought 470 sacks of corn seeds at 8 FRW per sack and bean seeds at 12 FRW per sack. For each 250 ha, 112 sacks of corn seeds are needed. If the total cost was 4260, how many sacks of bean seeds were bought?

4.2 Inequalities

4.2.1 Graphical representation of linear inequalities

So far, we have represented inequalities on a number line. In this section we are going to represent inequalities on a Cartesian plane. Remember that, (x, y) denotes any point on the Cartesian plane. Graph. 4.3 shows the graph of $x = 3$.



Graph 4.3

The line $x = 3$ divides the cartesian plane into three sets (3 regions) of points. These are:

(i) the set of point B on the line,

(ii) the set of point A on one side of the line i.e. to the left of the line.

(iii) the set of points C on the other side of the line i.e. to the right of the line.

The same line divides the plane into two regions A and C one on either side of the line.

Activity 4.3

Use Fig. 4.3 to do this activity.

Imagine that a Cartesian plane extends indefinitely in all directions and that a line also extends indefinitely in two directions in Fig. 4.3.

1. Into how many sets of points does the line $x = 3$ divide the Cartesian plane?
2. Identify and describe each set of points with reference to the line.
3. Into how many regions (areas) does the line divide the plane?

The x-co-ordinate for every point to the left of the line $x = 3$ is less than 3, i.e. $x < 3$.

For all the points to the right of line $x = 3$, the x-co-ordinate is greater than 3, i.e. $x > 3$.

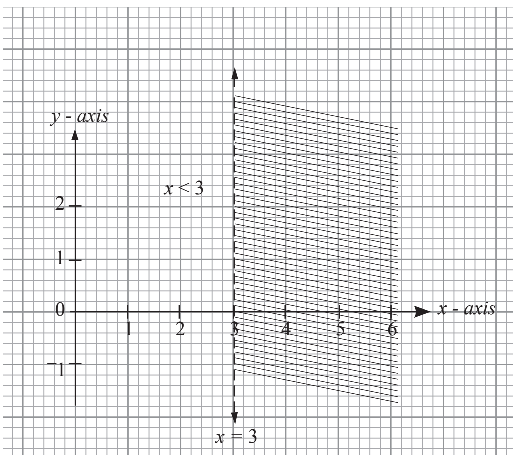
Graph 4.4 (a) shows the region containing all points (x , y) for which $x < 3$. This is shown with a dotted line meaning that points on the line are not part of the region.

Graph 4.4 (b) shows the region containing all points (x , y) for which $x \geq 3$. This is shown with a solid line, meaning that points on the line are part of the required region.

Note the following:

(a) Any line divides a plane into three sets of points i.e.
  - (i) Points on the line.
  - (ii) Points on either side of the line

(b) In a Cartesian plane, each set of points can be defined with reference to the line. For example, points on the line are defined by the equation of the line. Points on either side of the line can be described using the inequalities notation(s) $>$, $\geq$,$<$, $\leq$ with reference to the equation of the line. As shown in the graphs below.



Graph 4.4 (a)

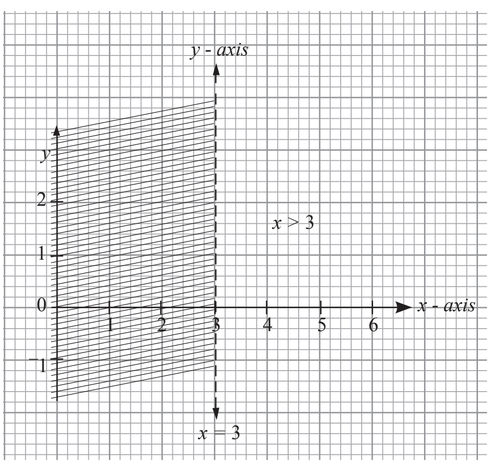


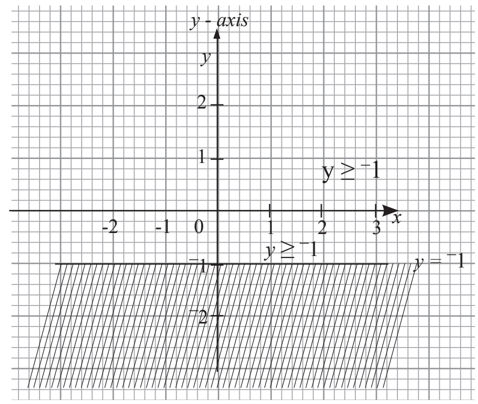
Fig. 4.4 (b)

Note:

In both cases, the **unwanted region** (i.e. the region in which the inequality is not satisfied) is shaded.

Also, on line $x = 3$ in graph. 4.4(a) are not wanted, so the line is 'dotted'.

Graph 4.5 shows the region for which $y \geq -1$.



Graph 4.5

Note that the line $y = -1$ is continuous. This means that the points on the line are included in the required region.

> To represent an inequality on a graph, we use the equation corresponding with that inequality as the equation of the boundary line;
>
> e.g. $y > -1$: Boundary line is $y = -1$.
>
> If points on the line are included in the required region, the line is continuous (solid). If not, the line is dotted (broken). The normal convention is to shade the unwanted region.

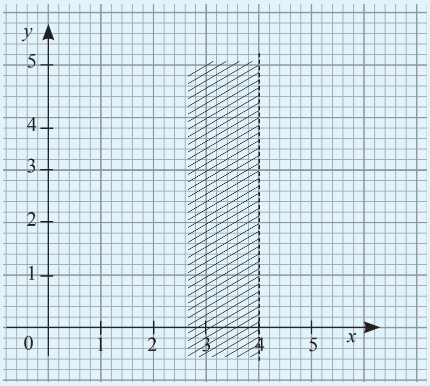
4.2.2 Forming inequalities from given regions

Points to note

To form inequalities from a given graph of inequalities, we use a step by step approach.

Activity 4.4

Use the figure given below to do Activity 4.4.



Graph 4.6

Step 1:

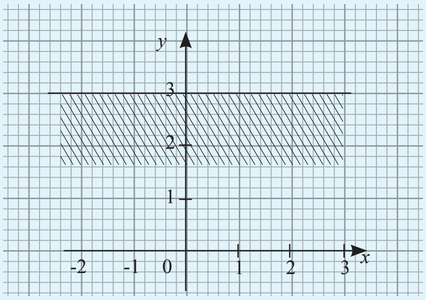
(i) Identify the line that defines the given region (boundary line).

(ii) Step 2: Find the equation of the boundary line.

(iii) Step 3: Identify the wanted region (unshaded region) and from it pick a point clearly **not** on the line.

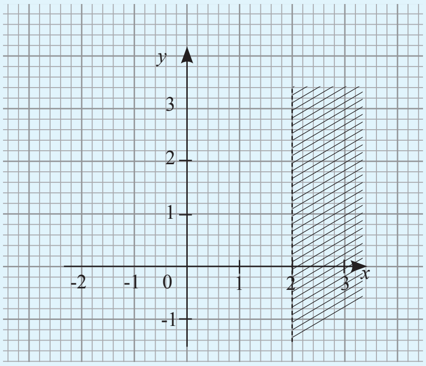
(iv) By substituting the coordinates in the equation in (iii) above you will be able to identify the inequality *t* satisfies.

(a)



Graph 4.7

(b)



Graph 4.8

(c)

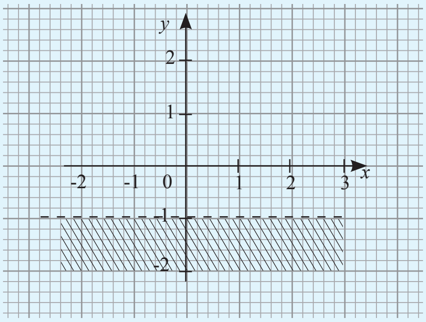


Fig. 4.9

The respective lines in Fig. 4.6 divide the Cartesian plane into two regions.

- Line (a) passes through the point with coordinates (0, 3). The line has the equation $y = 3$ and the points below the line are shaded. For all the points in the unshaded region, y value is greater than 3. Therefore, the region satisfies the inequality $y \geq 3$. The points on the line are also included since the line is continuous.
- Line (b) is not continuous and passes through the point (2, 0). The required region is to the left of the line $x = 2$. Therefore, the required region satisfies the inequality $x < 2$.
- Line (c) is broken and passes through (0, –2). The equation of the line is $y = –2$ and the required region is below

the line. Therefore, the required region satisfies the inequality $y < -2$.

If the required region is defined by a vertical line, the inequality will be of the form $x \le k$ or $x \ge k$ where k is a constant and line is solid.

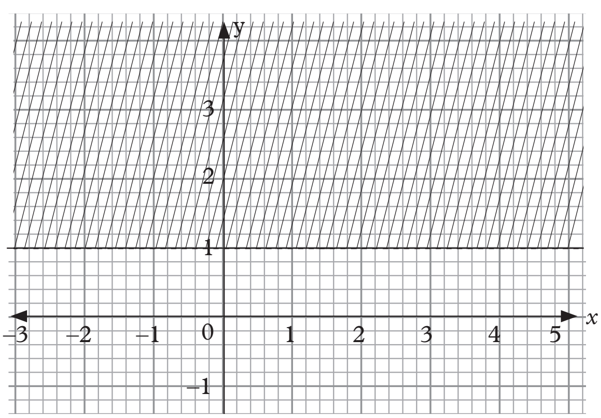
If the line is horizontal, the inequality will be of the form; $y \ge c$ or $y \le c$ where c is a constant. If line is broken, then the inequality will be $y > c$ or $y < c$.

Exercise 4.3

Show each of the following regions on a Cartesian graph.

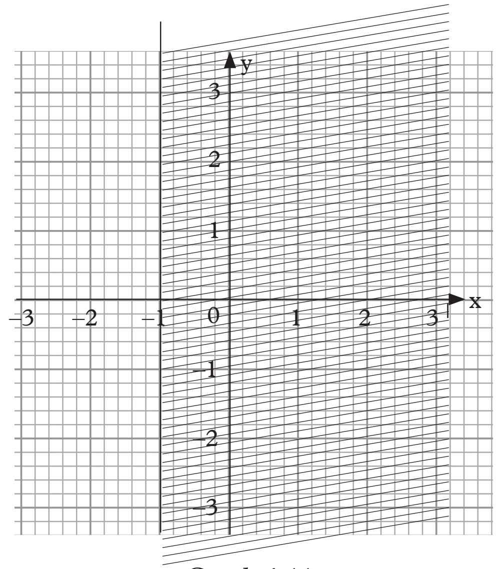
1. (a) $x > 1$ (b) $x < 5$
2. (a) $x < {}^{-}2$ (b) $x \ge {}^{-}1$
3. (a) $y \le 2$ (b) $y > 0$
4. (a) $y < \frac{1}{2}$ (b) $y \ge {}^{-}1.5$
5. $4x - x^2 \le x(1 - x) + 18$
6. Find the inequalities represented by the following unshaded regions Fig. 4.7.

(a)



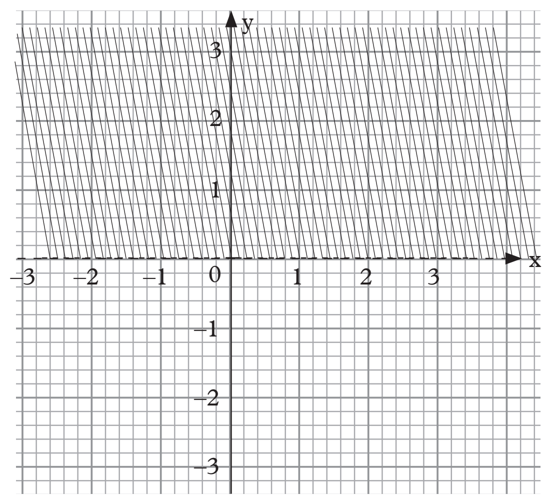
Graph 4.10

(b)



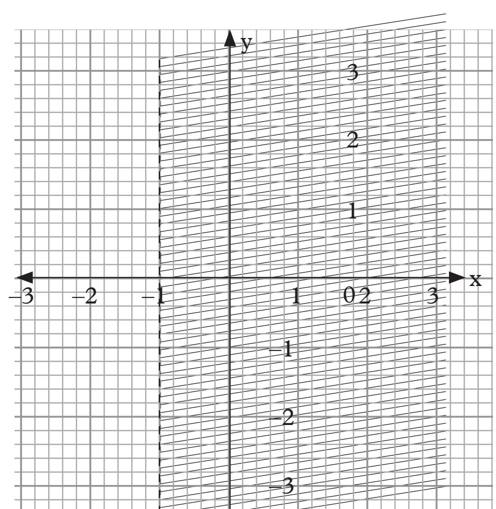
Graph 4.11

(c)



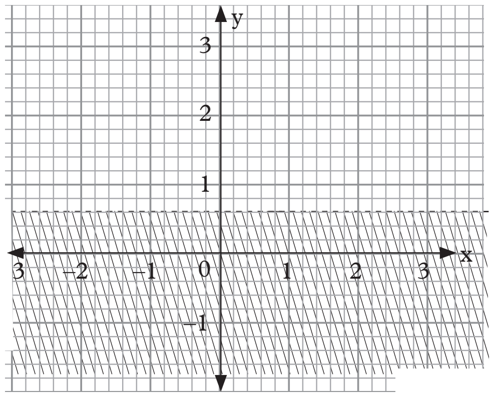
Graph 4.12

(d)



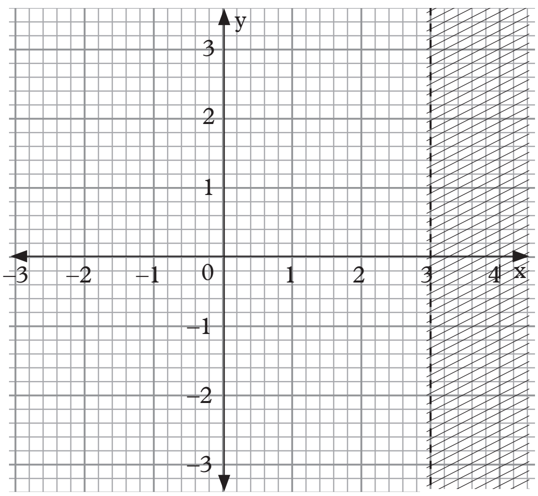
Graph 4.13

(e)



Graph 4.14

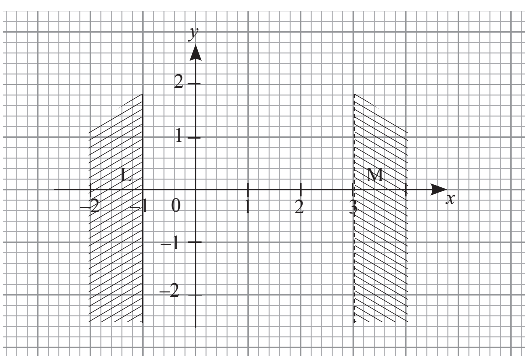
(f)



Graph 4.15

4.2.3 Simultaneous linear inequalities with one unknown

When regions are defined by two or more inequalities, those inequalities are referred to as simultaneous inequalities.



Graph 4.16

In Fig. 4.8, the unshaded region R lies between two inequalities. The two boundary lines are $x = -1$ and $x = 3$. The region R satisfies both the inequalities $x < 3$ and $x \geq -1$ simultaneously.

$\therefore$ We can say $-1 \leq x$ and $x < 3$.

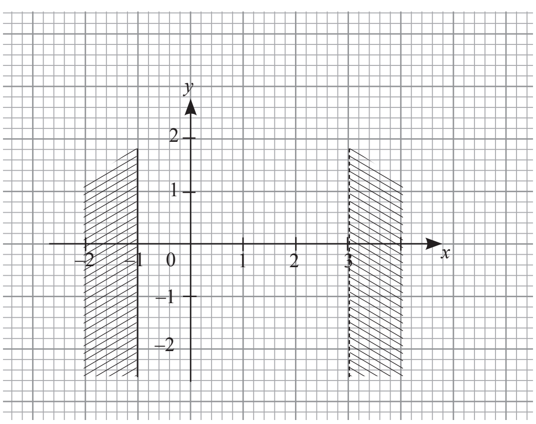
In one statement, we write $-1 \leq x < 3$.

Exercise 4.4

By shading the unwanted regions, show the regions which satisfy the given inequalities in questions 1 to 4.

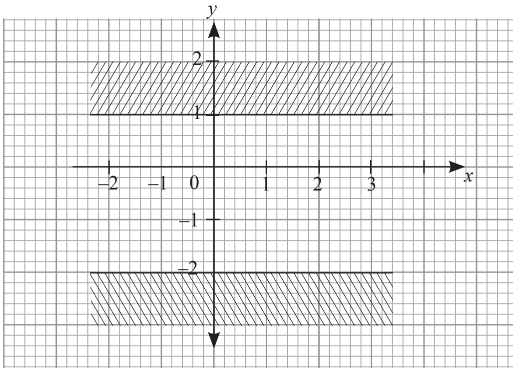
1. (a) $3 < x < 4$ (b) $1 < x \leq 5$
2. (a) $^{-}2 \leq y < 2$ (b) $^{-}1 \leq y \leq 1$
3. (a) $2 - \frac{1}{3}\ x \leq x \leq 4$
   (b) $2x > 7$ and $3x \leq 18$
4. $y + 5 \leq 4y < 2y + 14$
5. Find the inequalities represented by the following unshaded regions. Write your answer in a single statement, that is
   $a \leq x \leq b$ *or* $a \leq y \leq y \leq b$

(a)



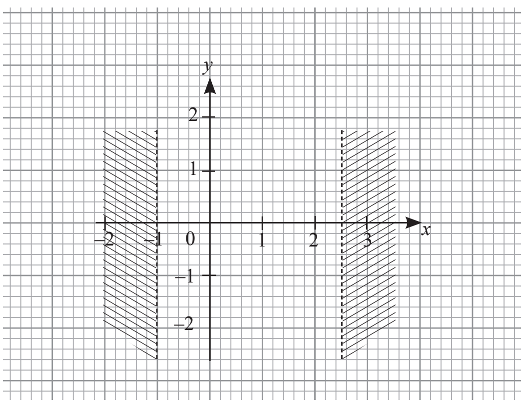
Graph 4.17

(b)



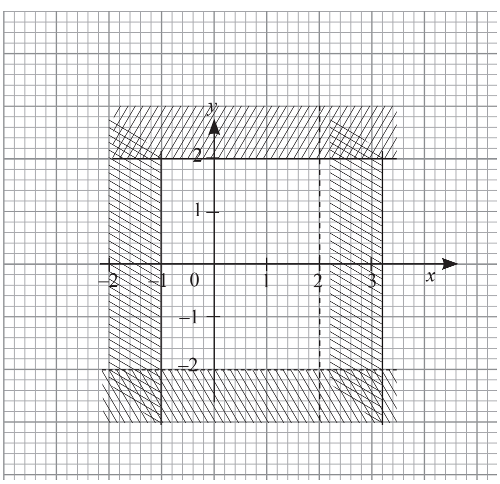
Graph 4.18

(c)



Graph 4.19

(d)



Graph 4.20

In (d) above, what is the name of the region represented by the inequalities?

In questions 6 to 8, show the regions satisfied by the given inequalities:

6. $^{-}1 < \frac{1}{2}x < 3$

7. $^{-}1 \leq 3x - 1 < 6$

8. $^{-}2 < y < 4$

4.2.4 Linear inequalities in two unknowns

We have dealt with inequalities of the form $x \leq a$, $x \geq b$, $y \geq c$, etc. where a, b and c are constants.

In this section, we look at inequalities of the form $ax + by \leq c$, $ax + by \geq d$, etc. where a, b, c and d are constants. Such an inequality is represented graphically by a region containing all the ordered pairs of values (x, y) which satisfy that inequality.

Activity 4.5

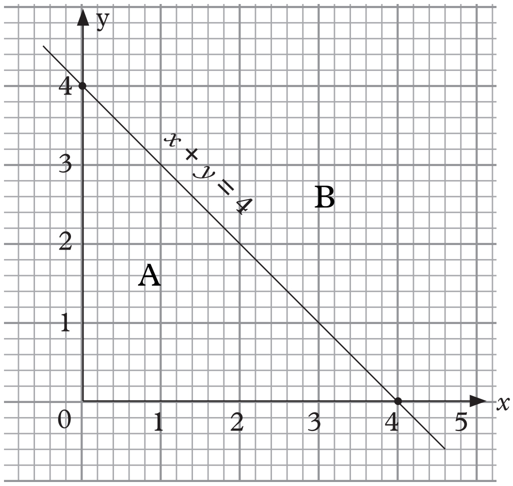
Using a graph paper, draw a line represented by the equation $2y = 3x + 6$

1. Into how many regions does the line divide the Cartesian plane?
2. Identify one point (x, y) on one side of the line and substitute for x and y in the given equation. What do you notice?

   Describe the region containing the point using inequality notation.
3. Do a similar substitution as in (2) above with a point from the other side of the line. Comment on your answer.

Now repeat Activity 4.5 above using the equation $x + y = 4$.

Graph 4.21 shows the graph represented by the equation $x + y = 4$.



Graph 4.21

(a) The line $x + y = 4$ divides the plane into two regions A and B

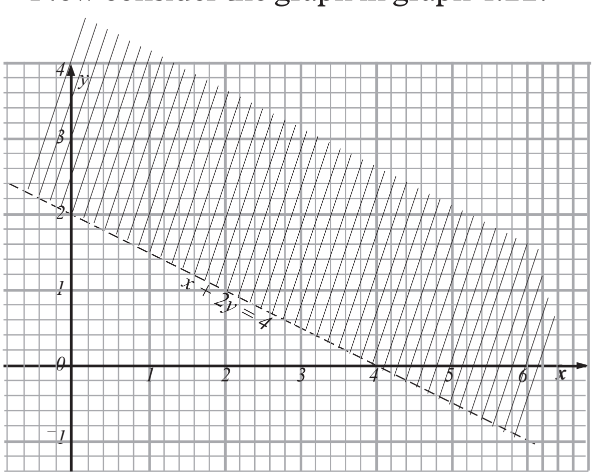
(b) Using a point such as (2,1) substitute in the equation, $x + y = 4$.
LHS = 2 + 1 = 3

Therefore (2, 1) satisfies $x + y < 4$

Thus $x + y$ is less than 4. The point (2, 1) in region A satisfies the inequality $x + y < 4$. But since the line is drawn solid, we can denote the region as $x + y \leq 4$.

Similarly, points on the other side satisfy the inequality $x + y \geq 4$.

Now consider the graph in graph 4.22.



Graph 4.22

The boundary line in graph 4.22 is broken.

This means unshaded (required) region does not include points on the line in order to define the required region, we use coordinates of any point not on the line.

For example use the point (1,1) substitute for x and y in the equation $x + 2y = 4$ as follows: in the equation $x + 2y = 4$,

LHS: $x + 2y = 1 + 2 \times 1$

$= 1 + 2 = 3$

RHS $= 4$

The value on the LHS is less than the value on the RHS, thus, $3 < 4$.

This means $x + 2y < 4$.

Therefore the unshaded region contains the points that satisfies the inequality $x + 2y < 4$.

Note: If the boundary line does not pass through the origin, it is more convenient and faster to determine the required region using the origin.

Exercise 4.5

Show the region which contain the set of points represented by each of the following inequalities in questions 1 to 4.

1. (a) $x + y < 4$ (b) $y - x < 4$
   (c) $2x + 3y \geq 6$
2. (a) $x - y \leq 0$ (b) $4x + 5y \leq 10$
   (c) $y + 3x > -6$
3. (a) $3x < y + 6$ (b) $\frac{1}{2}x - 2y > 2$
   (c) $y + \frac{5}{2}x \leq -5$
4. (a) $x + y \geq 3$ (b) $4y - 3x > 0$
   (c) $5y - x < 15$

4.2.5 Graphical solution of simultaneous linear inequalities with two unknowns

Activity 4.6

Using the same axis show the regions

(i) $x + y > 3$

(ii) $3x + 2y < 12$

1. In each case, find the equation of the boundary line.
2. Draw the line, one at a time.
3. In each case show the required region by shading the unwanted region.
4. Denote the unshaded region with letter R.
5. What can you say about this region?

When solving linear simultaneous equations, we look for values of the two unknowns that make the two equations true at the same time. Similarly, linear inequalities with two unknowns are solved to find a range of values of the two unknowns which make the inequalities true at the same time. The solution is represented graphically by a region.

To identify the required region, we deal with one inequality at a time to avoid confusion.

(i) Draw one line.

(ii) Identify the required region by shading the unwanted inequalities.

(iii) Repeat parts (ii) and (iii)for each of the given inequalities.

(iv) The unshaded region represents the set of the given inequalities. Example 4.3 below illustrates the procedure of solving simultaneous linear inequalities.

Example 4.3

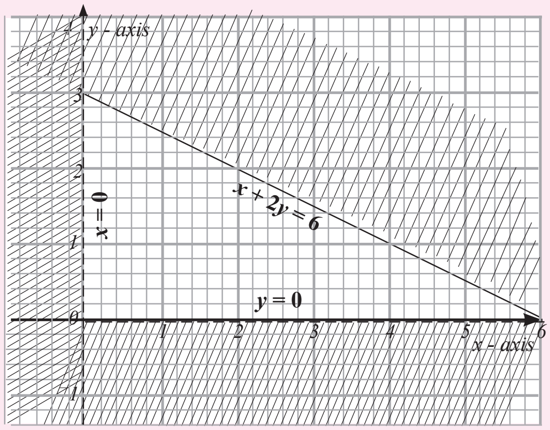
Draw the region which satisfies the following inequalities simultaneously:

$x > 0, y > 0, x + 2y \leq 6$

State the integral values of x and y that satisfy the inequalities.

Solution

In Fig. 4.12, $x + 2y = 6$ (solid line), $x = 0$ (broken line), $y = 0$ (broken line) are the boundary lines. All the ordered pairs of values (x, y) that satisfy the three inequalities lie within the unshaded region.



Graph 4.23

The integral pairs of x and y values that satisfy the inequalities are (1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (4, 1).

Exercise 4.6

For questions 1 to 5, show the region defined by all inequlities in each question.

1. $x + y \geq 0, x < 1, y > 1$
2. $2x + 3y > 6, y > 0, x > 0$
3. $y - x < 0, x < 4, y \geq 0$
4. $3x + 5y > 15, 5x + 3y < 30, x > 0, y > 0$
5. $y \geq 0, y < 4, 4x + 3y > 0, 5x + 2y < 15$
6. On the same graph, represent the solution of the simultaneous

inequalities.
$x < 7$, $y < 5$ and $8x + 6y \geq 48$

7. Use graphical method to solve the following inequalities simultaneously.
$x \geq 0$, $y \geq 0$, $x + y \leq 4$

8. Draw on the same diagram to show the regions representing the following inequalities.
$5x + 4y < 60$
$3x - y > -6$
$8x + 3y \geq 24$

9. Find the points with integral coordinates which satisfy the inequalities simultaneously
$x \leq 4$, $3y \leq x + 6$ and $2x + 3y > 6$

10. R is the region in a cartesian plane whose points satisfy the inequalities $0 \leq x < 5$, and $3 \leq 3y + x < 9$. Show R on the graph

11. On the same graph show the region that is satisfied by the inequalities
$x \geq 0$, $y \geq 0$, $x + y \leq 12$,
$x + 2y \leq 16$ and $y \geq \frac{-4}{3}x + 4$

12. Use graphical method to solve simultaneously the inequalities
$x \geq 0$, $y \geq 0$ and $x + y \leq 4$

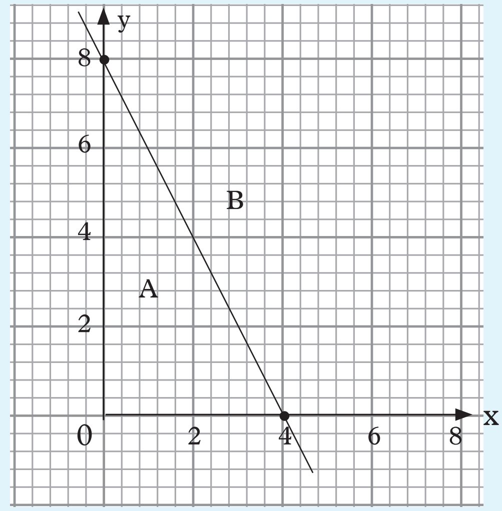
13. A region R is given by the inequalities $x \leq 6, y \leq 6, x + y < 9$ and $6x + 5y \geq 30$

Represent this region graphically and list all the points in the region which have integral coordinates.

4.2.6 Linear inequalities from inequality graphs

Activity 4.7

Consider the graph of line l in Graph 4.24

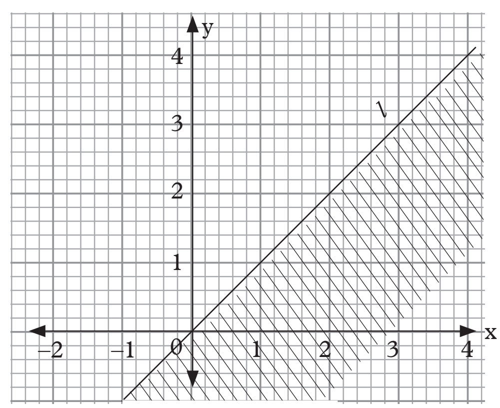


Graph 4.24

1. State the coordinates of at least five points on the line.
2. Use any two points to calculate the gradient of the line.
3. Find the equation of the line.
4. Describe the two regions A and B with reference to the line l.

Now consider graph 4.25.

Below, line L divides the Cartesian plane into two regions, the shaded and the unshaded.



Graph 4.25

In order to form an inequality from a given graph, we must be able to identify the line and find its equation first, and then proceed as per the following discussion:

(i) We find the equation of: line 1 that defines the region. For example, line 1 passes through many points (x, y). Using points such as (1, 1), (2, 2)…, we find gradient $= \frac{2-1}{2-1} = 1$
General equation of a line is given by $y = mx + c$ where m is the gradient and c is the y-intercept of the line. Thus $m = 1$, $c = 0$. Therefore the equation of the line is $y = x$.

(ii) Identify the required region. By convention, we shade the unwanted region. Thus we are interested in the unshaded region i.e. finding an inequality that describes the unshaded region.

(iii) Identify a point (x, y) on the plane, in the unshaded region, which is clearly not on the line i.e. (2, 3).

Substitute the values of x and y in the equation, one value at a time using equation $y = x$ and point (2, 3),

LHS = 3

RHS = 2

We see that the value on the LHS is greater than that of the RHS i.e. $3 > 2$.

This means that for all the points (x, y) in the unshaded region, $y > x$.

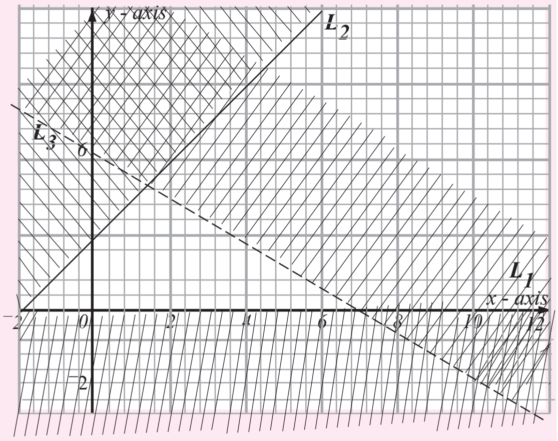
Since the line is drawn solid, it means points on the line should also be included in the required region.

Therefore, the required region is described by the inequality $y \geq x$.

Example 4.4 shows how one can find linear inequalities from inequality graphs.

Example 4.4

Write down the inequalities which are satisfied by the unshaded region in graph 4.26.



Graph. 4.26

Solution

Line L_1 is the x-axis i.e. $y = 0$ (solid line)

Points required are on the line or above it.

$\therefore$ *the inequality is $y \geq 0$*

Line L_2 is a solid line.

Let L_2 be $y = mx + c$.

Since L_2 intersects y-axis at $y = 2$ thus, $c = 2$.

Point (2, 4) is on line L_2. Substituting it in the equation of the line;

$y = mx + c$ *becomes* $4 = 2m + 2$

$\Rightarrow \quad m = 1$

$\therefore$ *Equation of L_2 is $y = x + 2$ or $y - x = 2$*

Point (2, 2) is on the wanted region. To know the inequality sign (i.e. < or >) we substitute

(2, 2) in the equation. Thus

$y = x + 2$ *becoms* $2 = 2 + 2$

The result shows $2 < 2 + 2$ thus, $y < x + 2$

Since L_2 is a solid line, the inequality is $y \leq x + 2$ or $y - x \leq 2$.

Equation of line L_3: $y = mx + c$

L_3 cuts the $y-$axis at $y = 6$, thus, $c = 6$. Also, point (10, 0) is on line L_3.

Hence, $y = mx + c$ *becomes* $0 = 10m + 6$

$\therefore m = -\frac{6}{10}$

$\therefore$ *Equation of* L_3 *is* $y = -\frac{6}{10}x + 6$

or $10y + 6x = 60$ *or* $5y + 3x = 30$

Point (2, 2) is on the wanted region.

To know the inequality sign, we substitute (2, 2) in the equation. Thus,

$y = \frac{-6}{10}x + 6$ *becomes,* $2 = \frac{-6 \times 2}{10} + 6.$

We get, $2 = \frac{-6}{5} + 6$ *which gives,* $2 = 4\frac{4}{5}$.

This shows that $2 < 4\frac{4}{5}$ *thus,* $y < \frac{-6}{10}x + 6$ *or*

$10y + 6x < 60$ *or* $5y + 3x < 30$.

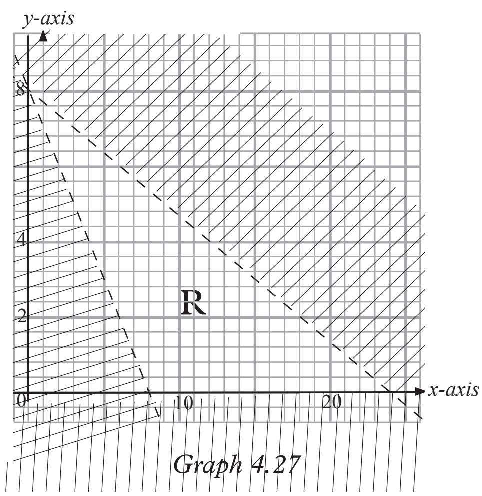
Since L_3 *is not solid, the inequality is* $10y + 6x < 60$ *or* $5y + 3x < 30$.

Thus the inequalities which satisfy the region are;

$y \geq 0,\ y - x \leq 2,\ 5y + 3x < 30.$

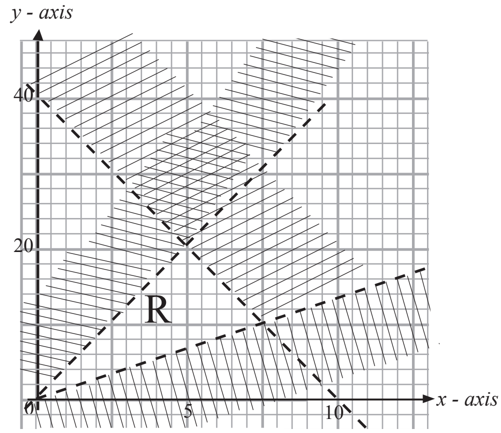
Exercise 4.7

1. Write down the inequalities satisfied by the region in graph. 4.27.



Graph 4.27

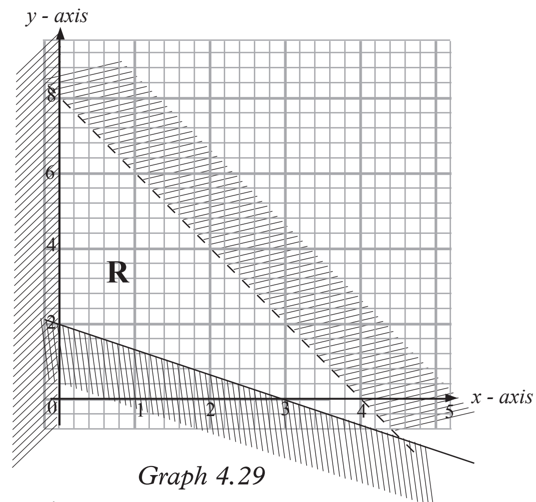
2. Write down the inequalities satisfied by the unshaded region R in Graph. 4.28.



Graph 4.28

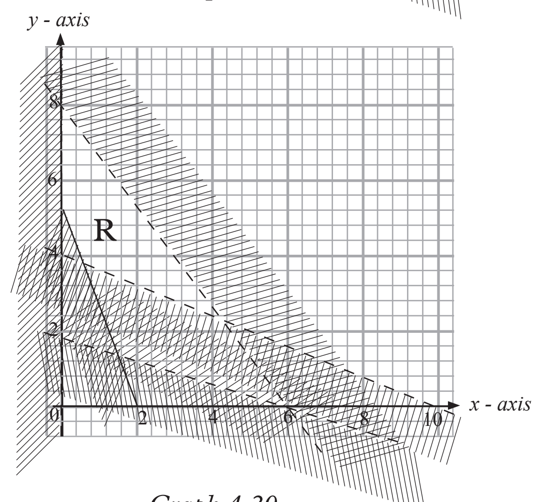
3. Use Graphs 4.29, 4.30 and 4.31 to find the equations of the boundary lines and hence find the inequalities that satisfy the unshaded regions.

(a)



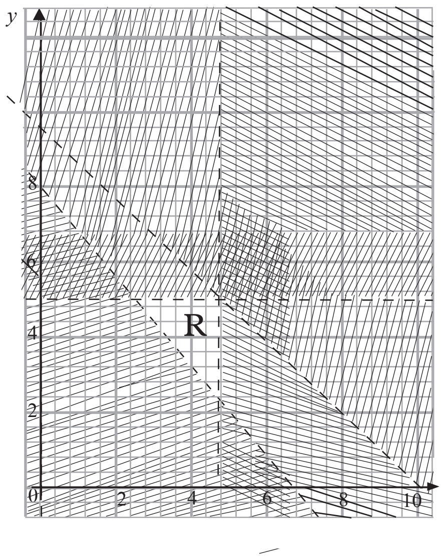
Graph 4.29

(b)

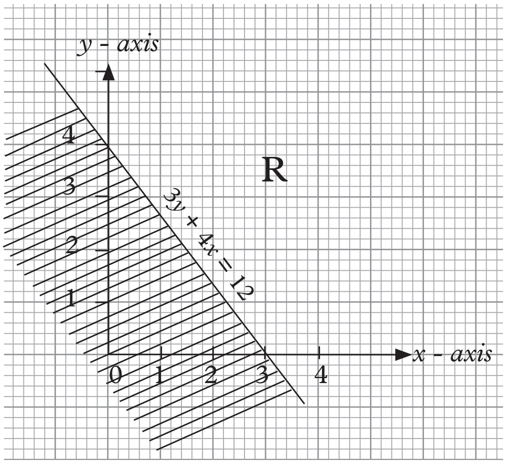


Graph 4.30

(c)

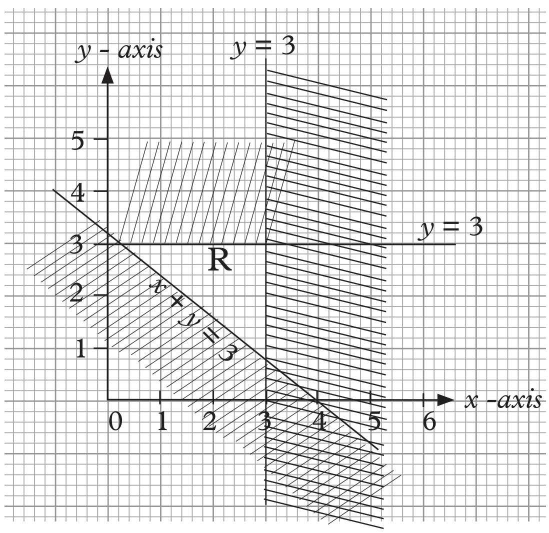


Graph 4.31



Graph 4.32

The unshaded region R satisfies the inequality $3y + 4x \geq$. This is the example of an open region.



The unshaded region R satisfies the inequalities $y \leq 3$, $x \leq 3$ and $x + y \geq 3$. R is an example of a closed region.

Remember:

- A line divides the Cartesian plane or any other plane into 2 regions.
- If also divides a plane into 3 sets of points i.e points on the line, and points on either side of the line each of which is defined by

Unit Summary

- To solve simultaneous equations graphically, we draw graphs of lines, representing the equations. If the equations have a unique solution, the lines will intersect at a point whose coordinates represent the solution set. If equations have no solutions, the lines will be parallel. If the equations have an infinite solutions set, the lines will be coincident. This means that any value of the variable you substitute will satisfy the equations.
- When forming simultaneous equations from a given situation, we begin by defining the variables we intend to use, then relate the two variables using the given information and solve the equations as recommended.
- Unlike in equations which have unique solutions, inequalities have a region for the solution set. The solutions may have closed or an open region.
  Remember:-

inequality rule. Remember the meaning of expressions such as $x<a$, $x\leq a$, $y>b$, $y\geq b$, $ax + by < c$ or $x > c$ with reference to regions in inequalities.

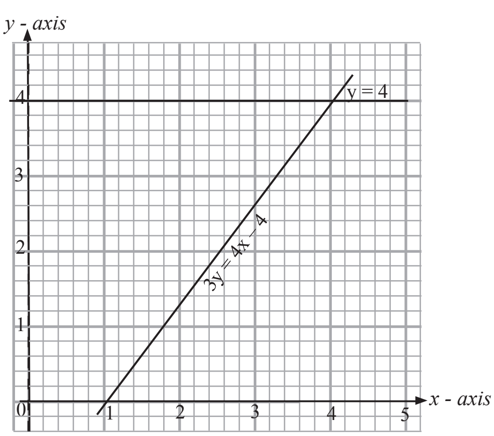
- Inequalities can be formed from given:
  - (i) inequality graphs
  - (ii) situations.

  Forming inequalities from graph:
  - (i) Identify the boundary line.
  - (ii) Find the equation of the line.
  - (iii) Using a point not on the line substitute the coordinates of the chosen point in the equation in order to determine the required region. Remember that by convention, we shade the unwanted region in order to leave the wanted region clean.

Unit 4 Test

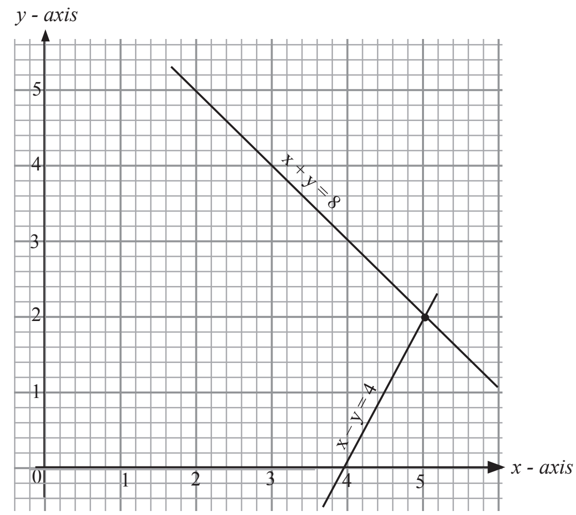
1. Find the solution of each of the equations, graphs represented in the following graphs.

(a)



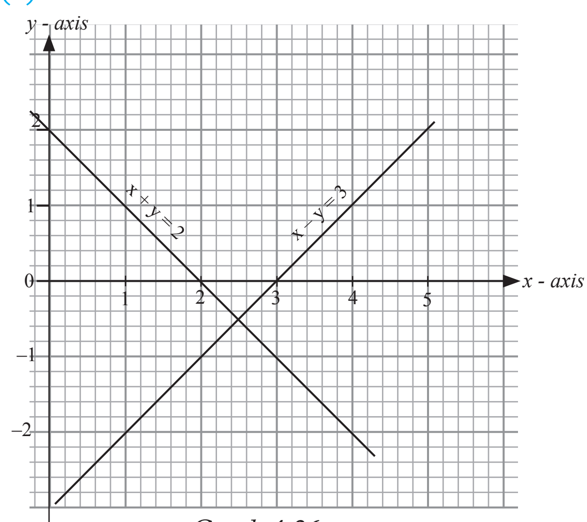
Graph 4.34

(b)



Graph 4.35

(c)



Graph 4.36

2. For which simultaneous equations is the ordered pair (3, –2) a solution?
   (a) $x + y = 1$, $x - y = 5$
   (b) $2x + y = 4$, $2y - x = -6$
   (c) $3x = 5 - 2y$, $x + y = 1$

3. Draw the graphs of $x + 2y = 8$ and $x - 2y = -4$ on the same axes. Use your graph to find the solution of the simultaneous equations $x + 2y = 8$ and $x - 2y = -4$.

4. Use graphical method to solve the following equations.

   (a) $x + 2y = 15$ (b) $2x = y - 5$

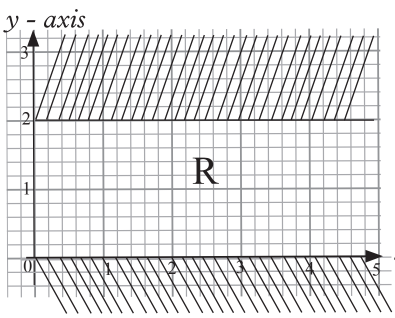
   $2x - y = 0$ $y = x - 3$

5. The sum of James' and David's ages is 34 years. Five years ago, the sum of twice James' age and three times David's age was 86 years. Using an appropriate variable for each age, form a pair of simultaneous equations and solve them to find the respective ages of the two boys.

6. Dan bought 5 boxes of sweets and 3 bags of candies for 1 205 FRW. If the cost of boxes and bags were reversed, the cost would have been 1 107 FRW. Find the cost of 1 box of sweets and 1 bag of candies.

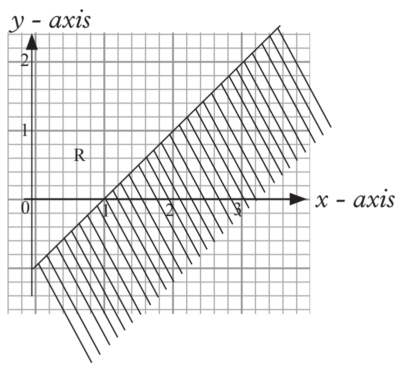
7. For each graph below write the inequalities that satisfy the unshaded region marked R.

(a)



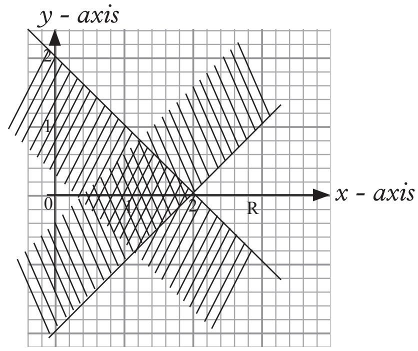
Graph 4.37

(b)



Graph 4.38

(c)



Graph 4.39

(d)

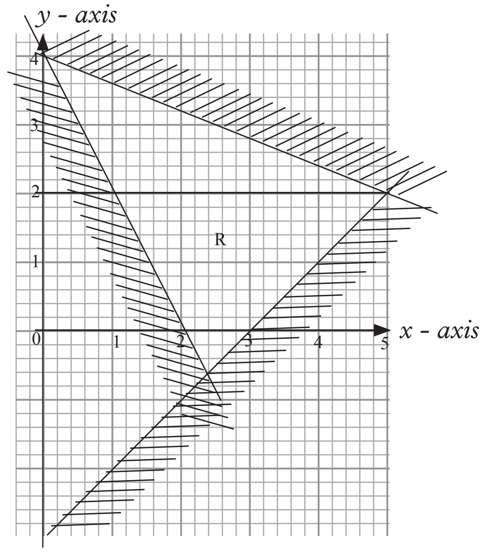


Fig. 4.40

8. (a) Draw the region given by the inequalities $x > 0$, $5x + 4y < 32$, $x + 2y > 10$.

   (b) State the coordinates of the vertices of the region in (a).

5 QUADRATIC EQUATIONS

Key unit competence: By the end of this unit, learners should be able to solve quadratic equations.

Unit outline

- Definition and examples of quadratic equations.
- Solving quadratic equations.
- Problems involving quadratic equations.

Introduction

Unit Focus Activity

1. Use your knowledge of solving equations to solve the following real- life problem.
   "A number of people planned to contribute a total of 800 000 FRW to start a small self help groups from where they could be getting small loans to boost their business. They agreed to contribute each an equal amount of money. Four of them withdrew from the venture. As a result, each of the remaining people had to contribute 10 000 FRW more to reach the target". Determine the:
   (a) Original number (x) of people in the venture.
   (b) Amount of money each contributed after the withdrawal of the four.
2. Compare your answer with those of other classmates during the class presentations.

Financial matters!

Pooling finances together for mutual benefit is one of the most successful ways of empowering one another financially. There is strength and security in numbers. Advice your working parents, siblings, friends and neighbours about it. Consider joining one as soon as you finish schooling.

5.1 Definition and examples of quadratic equations

Activity 5.1

1. Expand the following expressions.
   (a) $3(5x + 6)$
   (b) $(x - 5)(x - 2)$
   (c) $(x - 1)(x + 2)(x - 3)$
2. State the highest power of the unknown in each of the expression in (1) above.
3. Which of the expressions in question (1) above are quadratic?

A quadratic expression has the general form $\boldsymbol{ax^2 + bx + c}$ where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are real numbers. It is a polynomial of order 2.

Examples of quadratic expressions include $x^2 + 3x + 1$, $a^2 + 2a$, $y^2 - 2y + 1$, $c^2 - 4$ etc.

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers. Examples of quadratic equations are: $x^2 + 3x + 1 = 0$, $x^2 - 9 = 0$, $x^2 - 5x + 6 = 0$ *etc.*

Exercise 5.1

Expand each of the expression and state whether it is quadratic or not:

1. (a) $(x + 3)(x + 1)$
   (b) $(x + 2)(x - 4)(x - 1)$
   (c) $(x - 7)(x + 3)\ (x - 1)$

(d) $x(x - 4)(x - 3)$
(e) $(a^2 + 8)(a - 3)$
(f) $(p + 2)(p - 5)$

2. Which of the following belongs to quadratic expressions? Circle the correct expression.
   (a) $(x^3 - 5)(0 + 4)$
   (b) $(x + 5)(x + 4)$
   (c) $\frac{(x + 2)(5a + 3)}{x - 1}$
   (d) $(3x + 2)(x - 4)$
   (e) $(2x - 7)(4x - 3)$
   (f) $(x + 3)(x + 3)$
3. Write true/false in front of an expression if it is quadratic.
   (a) $(a - 4)(a^2 - 4)$
   (b) $6(x + 7)(x)$
   (c) $(5t + 3)(3t + 2)$
   (d) $(2 - x)(3 - x)$
   (e) $(3 + p)(5 - p)$
   (f) $(4 - 2y)(1 - 3y)$
   (g) $(3x + 1)(8 - 2x)$
   (h) $(5 - 3x)(2 - 4x)$
4. A man bought a certain number of shirts for 2 000 FRW. If each shirt had cost 200 FRW less, he could have bought five more for the same money.
   (a) Form and simplify an expression from the above information.
   (b) What name can be given to the above expression?
5. The perimeter of a rectangle is 42 cm. If the diagonal is 15 cm;
   (a) Form an expression that relates the perimeter, diagonal, length and width of the rectangle.
   (b) What special name is given to the expression in 5(a) above?

5.2 Solving quadratic equations

Quadratic equations are solved using the following methods:

- Factorisation method
- Graphical method
- Completing squares method
- Quadratic formula method

5.2.1 Quadratic equations by factorisation method

5.2.1.1 Factorising quadratic expressions

Activity 5.2

1. What do you understand by the term “factorise”?
2. Factorise the following quadratic expressions:
   (a) $x^2 + x$ (b) $4x^2 + 2x$
   (c) $x^2 - 4$
3. Factorise by grouping $x^2 + 2x + 3x + 6$. Hence. factorise $x^2 + 5x + 6$.
4. Using the fact you have learnt in 3 above, factorise $x^2 - x - 12$.

In order to find the factors of an expression such as $ax^2 + bx$, we look for the factors that are common in both terms. The common factors of ax^2 and bx is x.

Thus, $ax^2 + bx = x(ax + b)$.

Therefore, the factors of $ax^2 + bx$ are x and $(ax + b)$. This means, $ax^2 + bx$ can be factorised as $x(ax + b)$.

The process of finding the factors of an expression is called **factorisation**. This is the reverse of expansion.

Consider

$$(x + a)(x + b) = x(x + b) + a(x + b)$$
$$= x^2 + bx + ax + ab$$
$$= x^2 + (a + b)x + ab$$

The expressions $(x + a)$ and $(x + 6)$ are the factors of $x^2 + (a + b)x + ab$. Similarly:

$$(x + 3)(x + 4) = x(x + 4) + 3(x + 4)$$
$$= x^2 + 4x + 3x + 12$$
$$= x^2 + 7x + 12.$$

The last expression shows the relationship between the terms of the quadratic expression and its factors. Thus, a constant term ab of a quadratic expression is the product of the number terms a and b. Whereas, the coefficient of x is the sum of a and b. We can use these facts to factorise quadratic expressions.

Consider the expression $x^2 + 10x + 21$.

We can factorise it as follows;

Compare $x^2 + 10x + 21$ and $(x + \_\_)(x + \_\_)$, considering

$x^2 + (a + b)x + ab = (x + a)(x + b)$.

This means

$x^2 + 10x + 21 = x^2 + (a + b)x + ab$.

We need two numbers a and b such that $a + b = 10$ and $ab = 21$.

Clearly, a and b are factors of 21 whose sum is 10. These are 3 and 7.

Rewriting:

$x^2 + 10x + 21$ as $x^2 + (3 + 7)x + 21$

gives $x^2 + 3x + 7x + 21$.

$= x(x + 3) + 7(x + 3)$ *(factoring common terms)*

$= (x + 3)(x + 7)$.

Therefore, $x^2 + 10x + 21 = (x + 3)(x + 7)$.

Example 5.1

Factorise $x^2 + 7x + 12$.

Solution

We need two numbers a and b such that $a + b = 7$ and $ab = 12$. These numbers are 3 and 4.

Thus, $x^2 + 7x + 12 = x^2 + (3 + 4)x + 12$
$= x^2 + 3x + 4x + 12$
$= x(x + 3) + 4(x + 3)$
$= (x + 3)(x + 4)$

Example 5.2

Write an expression $x^2 - 6x + 8$ in terms of its factors.

Solution

Let, $x^2 - 6x + 8 = (x + a)(x + b)$.

Thus. $x^2 - 6x + 8 = x^2 + (a + b) + ab$.

Then, $(a + b) = -6$ *and* $ab = 8$.

Two numbers which add up to – 6 and whose product is 8 are –2 and –4.

So, $x^2 - 6x + 8 = x^2 + (-2 + -4)x + 8$
$= x^2 - 2x - 4x + 8$
$= x(x - 2) - 4(x - 2)$
$= (x - 2)(x - 4)$

Example 5.3

Factorise:

(a) $x^2 + 12x + 36$ (b) $x^2 - 8x + 16$

Solution

(a) *Let* $x^2 + 12x + 36 = (x + p)(x + q)$
$= x^2 + (p + q)x + pq$

Thus, $p + q = 12$ *and* $pq = 36$.

The factors of 36 whose sum is 12, are 6 and 6.

So, $x^2 + 12x + 36 = (x + 6)(x + 6)$
$= (x + 6)^2$

(b) Let, $x^2 - 8x + 16 = (x + p)(x + q)$

$= x^2 + (p + q)x + pq$

Thus, $p + q = -8$ and $pq = 16$

The factors of 16 whose sum is –8, are –4 and –4.

So, $x^2 - 8x + 16 = (x - 4)(x - 4)$

$= (x - 4)^2$

Note: $x^2 + 16x + 36$ and $x^2 - 8x + 16$ are perfect squares because their factors are identical.

When factorising a quadratic expression of the form $ax^2 + bx + c$ we get two factors whose sum is b and the product is ac.

Consider a quadratic expression $6x^2 + 5x - 4$. It can be factorised as follows:

Comparing $6x^2 + 5x - 4$ to $ax^2 + bx + c$, we find out that $a = 6, b = 5$ and $c = -4$.

We then get two values whose product is –24 and whose sum is 5 ie $ac = -24$ and $a + b = 5$. These values must be the factors of –24 and they are –3 and 8.

We then write

$$6x^2 + 5x - 4 = 6x^2 + (8 - 3)x - 4$$
$$= 6x^2 + 8x - 3x - 4.$$

By factorizing out the common factors,

We get

$$6x^2 + 8x - 3x - 4 = 2x(3x + 4) - 1(3x + 4).$$

So, $6x^2 + 8x - 3x - 4 = (2x - 1)(3x + 4)$

Example 5.4

Write down $7 + 3x^2 - 22x$ in terms of its factors.

Solution.

Rearranging the terms,

$7 + 3x^2 - 22x = 3x^2 - 22x + 7$

We have the coefficients $a = 3$, $b = -22$ and $c = 7$.

Sum is -22 and product is 21. The two factors that can add to give –22 and multiply to give 21 are –21 and –1.

We write $3x^2 - 22x + 7 = 3x^2 - 21x - x + 7$.

By factorising,

$3x^2 - 21x - x + 7 = 3x(x - 7) - 1(x - 7)$

$= (3x - 1)(x - 7)$

Exercise 5.2

1. Factorise the following expressions:
   (a) $x^2 + 6x + 8$
   (b) $x^2 - 8x + 15$
   (c) $x^2 + 2x - 35$
   (d) $x^2 - 9$
   (e) $x^2 - 49$
   (f) $x^2 + x - 12$
2. Can the following expressions factorise ? Write YES or NO for the results.
   (a) $x^2 + 9x + 14$
   (b) $x^2 - 11x - 12$
   (c) $5u^2 + 6u + 9$
   (d) $2t^2 - t - 42$
3. A man bought a certain number of shirts for 2 000 FWR. If each shirt had cost 200 FWR less, he could have bought five more for the same money.
   (a) Form and simplify an expression from the above information.
   (b) What name can be given to the above expression?
   (c) Factorise the expression obtained in 3(b) above.
4. The perimeter of a rectangle is 70 cm. If the diagonal is 25 cm:
   (a) Form an expression that relates the perimeter, diagonal, length and width of the rectangle.
   (b) Factorise the expression obtained in 4(a) above.

5.2.1.2 Solving quadratic equations by factorisation method.

Activity 5.3

1. Given that $(x-2)(x+3)=0$, what are the possible values of $(x-2)$ and $(x+3)$? Obtain the values of x for each expression.
2. Given that $(x+4)(x+6)=0$, find two possible values of x.
3. Factorise completely $4x^2-2x$. Hence find the values of x given that $4x^2-2x=0$.

(Hint: Use the idea learnt above.)

In order to solve a quadratic equation, the quadratic expression is factorized so that the equation is in the form $(x+a)(x+b)=0$.

Then either $(x+a)=0$ or $(x+b)=0$.

Thus $x=-a$ and $x=-b$.

Example 5.5

Solve: $(x-4)(x+1)=0$

Solution

If $(x-4)(x+1)=0$*, then either* $x-4=0$ *or* $x+1=0$.

Therefore, $x=4$ *or* $x=-1$.

Hence the roots of the equation $(x-4)(x+1)=0$ *are 4 or –1.*

Example 5.6

Solve the equation $x^2+7x+6=0$.

Solution

$x^2+7x+6=0$

Factorizing, x^2+7x+6*, gives* $(x+6)(x+1)=0$.

Therefore, $x+6=0$ *or* $x+1=0$;

Which means $x=-6$ *or* $x=-1$

Example 5.7

Solve: $x^2-x-29=1$

Solution

Always ensure that the quadratic expression is equated to zero. This is the only time the method used in these examples can apply.

Thus, $x^2-x-29=1$ *should be rewritten as* $x^2-x-29-1=0$*. That is,*

$x^2-x-30=0$.

The factors of –30, whose sum is –1 and product –30, are –6 and 5.

We write $x^2-6x+5x-30=0$

We then factorise and get,

$x(x-6)+5(x-6)=(x-6)(x+5)=0$.

Either $x-6=0$ *or* $x+5=0$.

$x=6$ *or* $x=-5$.

The roots are –5 or 6.

Example 5.8

Solve:

(a) $x^2-49=0$ (b) $x^2-6x=0$

Solutions

(a) $x^2-49=0$ *can be written as*

$x^2-7^2=0$

$(x-7)(x+7)=0$

$x-7=0$ *or* $x+7=0$

$x=7$ *or* $x=-7$.

The roots are -7 or 7.

(b) *Factorizing* $x^2-6x=0$ *gives*

$x(x-6)=0$

Either $x=0$ *or* $x-6=0$

$x=0$ *or* $x=6$

The roots are 0 or 6.

Example 5.9

Solve the quadratic equation.

$6x^2+5x-4=0$

Solution

$6x^2 + 5x - 4 = 0$

When the coefficient of x^2,(in this case it is 6), in the quadratic expression is numerically greater than 1, we proceed as follows when factorising:

- *Multiply the coefficient of x^2 by the constant term, i.e. $6 \times -4 = -24$.*
- *Find the factors of –24 whose sum is 5, (the coefficient of x), i.e. –3 and 8.*
- *Rewrite the equation as:*

  $6x^2 + (-3 + 8)x - 4 = 0$

  $\Rightarrow 6x^2 - 3x + 8x - 4 = 0$

  $\Rightarrow 3x(2x - 1) + 4(2x - 1) = 0.$

  $\Rightarrow (2x - 1)(3x + 4) = 0.$

  Then, either $2x - 1 = 0$ or $3x + 4 = 0$

  $\Rightarrow 2x = 1$ or $3x = -4$

  $x = \frac{1}{2}$ or $x = -\frac{4}{3}$.

Example 5.10

Write down $7 + 3x^2 - 22x$ in terms of its factors. Hence solve $7 + 3x^2 - 22x = 0$.

Solution

Rearranging the terms, $7 + 3x^2 - 22x = 3x^2 - 22x + 7$.

We have the coefficients $a = 3$, $b = -22$ and $c = 7$

Sum is –22 and product is 21. The two factors that can add to give –22 and multiply to give 21 are –21 and –1.

We write $3x^2 - 22x + 7 = 3x^2 - 21x - x + 7$

By factorising, $3x^2 - 21x - x + 7$

$= 3x(x - 7) - 1(x - 7)$

$= (3x - 1)(x - 7)$

Hence, $3x^2 - 22x + 7 = (3x - 1)(x - 7)$.

For solving,

The equation $7 + 3x^2 - 22x = 0$ after factorisation is written as

$(3x - 1)(x - 7) = 0.$

Either $3x - 1 = 0$ or $x - 7 = 0$

$3x = 1$ or $x = 7.$

$x = -\frac{1}{3}$ or $x = 7.$

Exercise 5.3

1. Obtain the values of x from the following expressions.
   (a) $(x + 4)(x + 2) = 0$
   (b) $(x - 5)(x - 3) = 0$
   (c) $(x - 5)(x + 7) = 0$
   (d) $(x^2 - 9) = 0$
2. Factorise the following expressions hence find the values of unknowns.
   (a) $x^2 + 9x + 14 = 0$
   (b) $x^2 - 11x - 12 = 0$
   (c) $u^2 + 6u + 9 = 0$
   (d) $t^2 - t - 42 = 0$
   (e) $a^2 - 2a + 1 = 0$
   (f) $y^2 + 8y = 0$
3. Can the following equations be solved by factorisation? Write True/ False and write down the solution set of equation.
   (a) $x^2 + 10x = 24$
   (b) $x^2 = 4x - 3$
   (c) $6x^2 - 29x + 35 = 0$
   (d) $6x^2 - x + 1 = 0$

5.2.2 Solving quadratic equations by graphical method

Activity 5.4

Complete the following tables of values for;

1. (a) Function $y = 2x^2 + 3x - 2$.

x	–3	–2	–1	0	1	2
y						

Table 5.1

(b) Function $y = x^2 + 3x + 6$.

x	-3	-2	-1	0	1	2
y						

Table 5.2

2. State the coordinates in 1 (a) and 1(b) in ordered pairs (x, y).
3. Plot the graphs for 1(a) and 1(b) on different Cartesian planes.
4. For each graph, read and record the x-coordinate(s) of the point(s) where the graph cuts x-axis.
5. Solve the equations $2x^2 + 3x - 2 = 0$ and $x^2 + 3x + 6 = 0$ by factorisation method.

In a quadratic function graph, the x-coordinate of the point where the graph cuts x-axis gives the solution to the quadratic equation represented by the function.

1. When the graph cuts the x-axis at one point, then the equation has one repeated solution.
2. When the graph cuts x-axis at two points, then the equation has two different solutions.
3. When the graph does not cut x-axis at any point, then the equation has no solution in the field of real numbers.

Example 5.11

Draw the graphs of $y = x^2$ and $y = -x^2$ for values of x between -2 and $+2$.

Solution

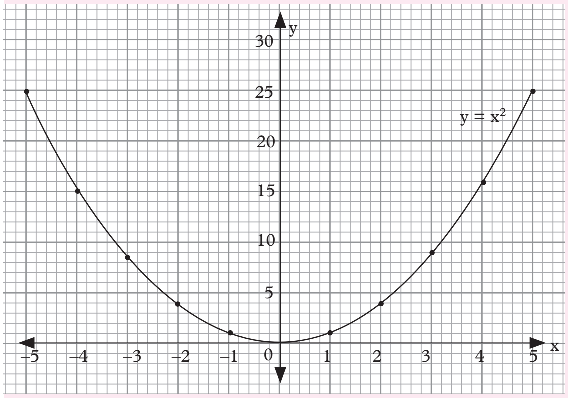
Graph of $y = x^2$:

make a table of values for (x, y)

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$y = x^2$	25	16	9	4	1	0	1	4	9	16	25

Table 5.3

Plot these points on a graph paper and label the axes x and y. Since the values of y are all positive, x-axis should be drawn near the bottom of the paper.



Graph 5.1

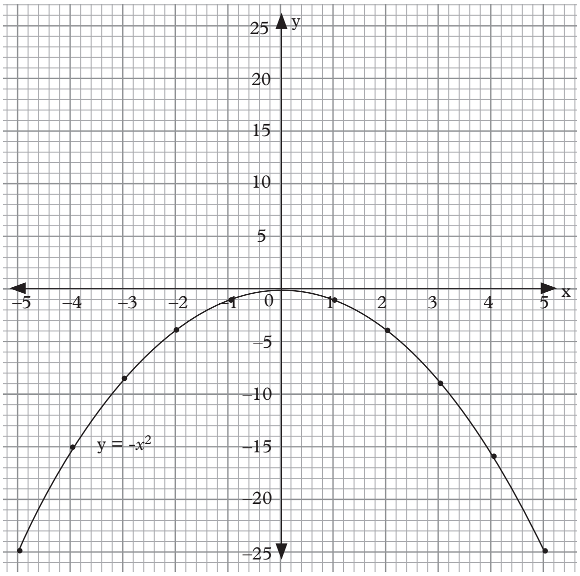
The equation $x^2 = 0$ has only one repeated solution since the graph cuts x-axis at only one point i.e. $x = 0$.

The graph of $y = -x^2$

All values of y are numerically the same as the corresponding values of y in $y = x^2$ but are **negative**. The shape of the curve will be the same but inverted.

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$y = -x^2$	–25	–16	–9	–4	–1	0	–1	–4	–9	–16	–25

Table 5.4



Graph 5.2

The equation $x^2 = 0$ has only one repeated solution since the graph cuts x-axis at only one point i.e. x = 0.

Example 5.12

(a) *Plot the graph of $y = x^2 + 3x - 4$ for values of x from –5 to +2.*

(b) *Use the graph to find:*

(i) *The value of y when x = 1.5*

(ii) *The value of x when y = 4*

(iii) *The value of y when x = –2.5*

(c) *Use the graph to solve the quadratic equation $x^2 + 3x - 4 = 0$.*

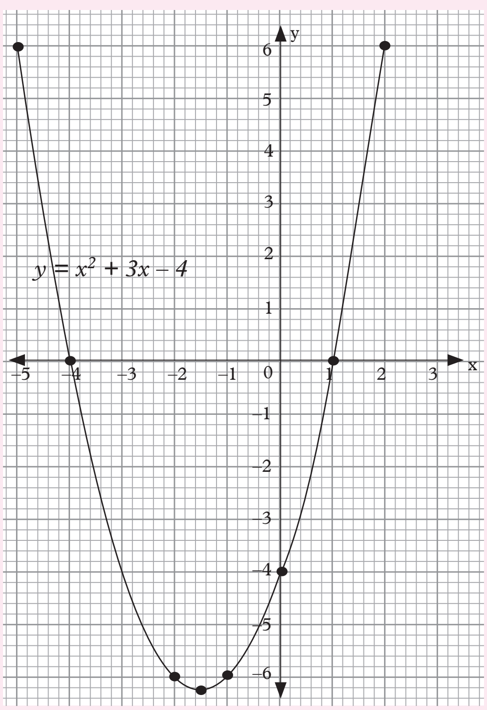
Solution

(a) *Table of values*

x	–5	–4	–3	–2	–1	0	1	2
x^2	25	16	9	4	1	0	1	4
$3x$	–15	–12	–9	–6	–2	0	2	4
-4	–4	–4	–4	–4	–4	–4	–4	–4
$y = x^2 + 3x - 4$	6	0	–4	–6	–6	–4	0	6

Table 5.5

Points to plot (–5, 6), (–4, 0), (–3,–4), (–2,–6), (–1,–6), (0,–4), (1, 0), (2, 6)



Graph 5.3

(a) *See the graph above,*

(b) *(i)* *y = 2.7*

(iii) *x =1.7 or – 4.7*

(iv) *y = –5.2*

(c) *The values of x are obtained at points where the curve meets the x-axis. i.e. x = – 4 and x = 1*

Example 5.13

(a) *Plot the graph of $y = x^2 - 3x + 2$, for values of x between -1 and +4 (include 1.5 in your table of values).*

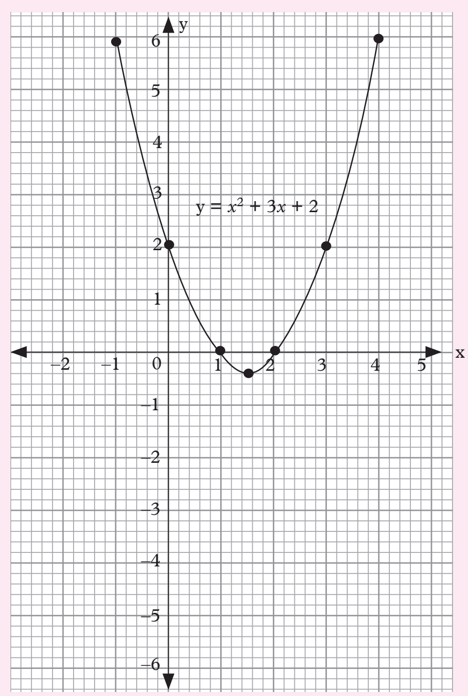
(b) *Use the above graph to solve $x^2 - 3x + 1 = 0$.*

Solution

Make a table of values of x and y:

x	–1	0	1	1.5	2	3	4
x^2	1	0	1	2.25	4	9	16
$-3x$	3	0	–3	–4.5	–6	–9	–12
+2	+2	+2	+2	+2	+2	+2	+2
$y = x^2 - 3x + 2$	6	2	0	–0.25	0	2	6

Table 5.6



Graph 5.4

Fig. 5.4 is the graph of function $y = x^2 - 3x + 2$. It crosses the x-axis at points where $x = 1$ and $x = 2$, i.e. These are the points on the curve for which $y = 0$.

Therefore $x = 1$ and $x = 2$ are the solutions of $x^2 - 3x + 2 = 0$.

(b) *Equation to be solved is $x^2 - 3x + 1 = 0$*

i.e. $x^2 - 3x = -1$

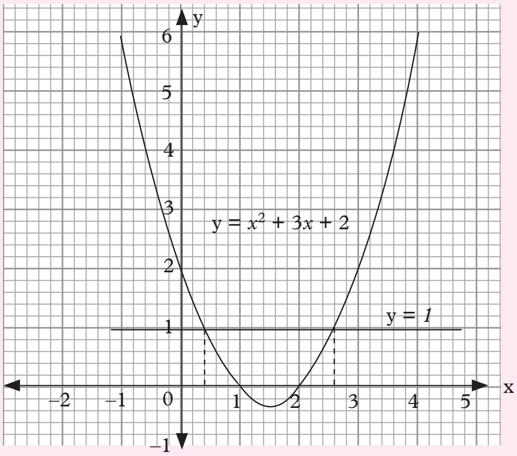
Substituting $x^2 - 3x = -1$ in the equation of the curve $y = x^2 - 3x + 2$,

We get $y = (x^2 - 3x) + 2$

$y = -1 + 2$

$y = 1.$

We need the points on the curve when $y = 1$. Draw the line $y = 1$ on the same axes.



Graph 5.5

The values of x at the points of intersection of the line and the curve are the solution of the equation.

From the graph, values of x are 0.4 and 2.6.

Example 5.14

(a) *Draw the graph of $y = 1 + x - 2x^2$, taking values of x in the Domain $-3 < x < 3$.*

(b) *Using the same scale and axes, draw the graph of $y = x - 5$.*

(c) *Use your graphs to answer the following questions:*

(i) *Write down the x-coordinates of the points of intersection of the functions*

$y = 1 + x - 2x^2$ and $y = x - 5$.

(ii) *Show that these values of x satisfy the equation $2x^2 + x - 6 = 0$.*

Solution

(a) *Table of values for $y = 1 + x - 2x^2$*

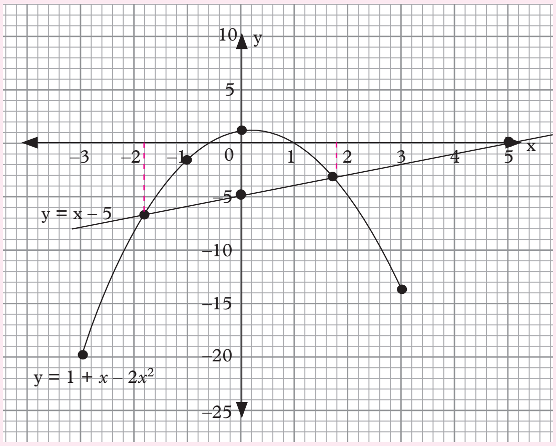
x	–3	–2	–1	0	1	2	3
1	1	1	1	1	1	1	1
x	–3	–2	–1	0	1	2	3
$-2x^2$	–18	–8	–2	0	–2	–8	–18
y	–20	–9	–2	1	0	–5	–14

Table 5.7

Table for $y = x - 5$

x	0	3
y	–5	2

Table 5.8



Graph 5.6

(b) *When plotted, a curve and a straight line as shown in the figure below are obtained.*

(c) *From the graph,*

(i) *Values of x where the straight line intersects the curve are $x = -1.8$ and $x = 1.8$.*

(ii) At these points of intersection, $1 + x - 2x^2$ and $x - 5$ are equal.

Hence $1 + x - 2x^2 = x - 5$

$1 + 5 + x - x - 2x^2 = 0$

$6 - 2x^2 = 0.$

Or $2x^2 + x - 6 = 0$

$2x^2 = 6$

$x^2 = 3$

$x = \pm\sqrt{3}$

So, $x = -1.8$ *or* 1.8.

Exercise 5.4

1. Plot the graph of $y = x^2 - 4x + 4$ for values of x from -1 to +5. Solve from your graph the equations:
   (a) $x^2 - 4x + 4 = 0$
   (b) $x^2 - 4x + 1 = 0$
   (c) $x^2 - 4x - 1 = 0$.
2. Plot the graph of the function $x^2 - 6x + 5$ for $-1 < x < 7$. Use your graph to solve the equation $x^2 - 6x + 5 = 0$.
3. Draw the graph of $y = 2x^2 - 7x - 2$ for values of x from -3 to +3. Use your graph to solve the equations:
   (a) $2x^2 - x = 4$,
   (b) $2x^2 - x + 6 = 0$
   (c) $2x^2 - x - 4 = 2x$.
4. Plot the graph of $y = \frac{1}{4}x^2$ for domain $-4 < x < 4$. Using the same scale and axes, draw the curve of $y = \frac{1}{4}x^2 - 3$.
5. Plot the graph of the function $y = 2 - x - x^2$ from domain $-3 < x < 3$. Use your graph to solve the equation $x^2 + x = 2$.
6. Draw the graph of the function $y = 2x^2 - 7x - 2$ for values of x from -1 to 5. By drawing suitable lines on the same axes, solve, where possible, the following equations:
   (a) $2x^2 - 7x = 2$,
   (b) $2x^2 - 8x + 4 = 0$
   (c) $2x^2 - 7x + 7 = 0$.
7. Copy and complete the following table of values for $y = 6 + 3x - 2x^2$.

x	-2	-1	0	0.5	1	2	3
$3x$							
6							
$-2x^2$							
y	-8				7	4	

Table 5.9

Draw the graph of $y = 6 + 3x - 2x^2$ for domain $-2 < x < 3$.

Use the graph to obtain solutions of the equations:

(a) $6 + 3x - 2x^2 = 0$
(b) $2 + 3x - 2x^2 = 0$
(c) $3 + x - 2x^2 = 0$.

8. Plot the graph of $y = x^2 - x - 2$ after completing the following table for values of x and y.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
x^2											
$-x$											
-2											
y		1.75	0	-1.25				-1.25		1.75	

Table 5.10

By drawing a suitable straight line on your graph, solve the equation $x^2 - 2x = 0$.

9. Draw the graph of $y = \left(\frac{1}{2}x - 3\right)(x + 3)$ for domain $-3 < x < 4$.

From your graph find the values of x for which $(\frac{1}{2}x - 3)(x + 3) = 2$.

10. Plot the graph of the function $y = x^2 - 3x$ for domain $-1 < x < 4$. Use your graph to find:
    (a) The range of values of x for which the function is negative.

(b) The solutions of the equations $x^2 - 3x = 1$ and $x^2 - 2x - 1 = 0$.

11. Given that $y = (3x + 1)(2x - 5)$, copy and complete the following table for values of x and y.

x	–1	–0.5	0	0.5	1	1.5	2	2.5	3	3.5	4
y	14	3	–	–	–12	–11	–	–	–	23	–

Table 5.11

Plot the graph of $y = (3x + 1)(2x - 5)$ from $x = -1$ to $x = 4$.

By considering the points of inter-section of the two graphs, a certain quadratic equation in x can be solved. Write down and simplify the equation and obtain its roots from the graphs.

5.2.3 Solving quadratic equations by completing squares method

5.2.3.1 Perfect squares

Activity 5.5

Determine whether the following quadratic expressions are perfect squares or not.

(a) $x^2 + 8x + 16$

(b) $x^2 + 10x + 12$

(c) $9x^2 + 6x + 1$

(d) $36x^2 - 8x + 4$

Recall that:

1. $(a + b)^2 = a^2 + b^2 + 2ab$, and
2. $(a - b)^2 = a^2 + b^2 - 2ab$

The right hand sides of 1 and 2 are the expansions of the squares on the left hand sides. Such expressions are called **perfect squares**. They may be written in words as:

1. The square of the **sum** of two numbers is equal to the sum of their squares **plus** twice their product.
2. The square of the **difference** of two numbers is equal to the sum of their squares **minus** twice their product.

Example 5.15

Write down the square of:

(a) $4x + y$ (b) $2x - 3y$

Solution

(a) $(4x + y)^2 = (4x)^2 + (y)^2 + 2(4x)(y)$
$= 16x^2 + y^2 + 8xy$

(b) $(2x - 3y)^2$
$= (2x)^2 + (3y)^2 - 2(2x)(3y)$
$= 4x^2 + 9y^2 - 12xy$

After a little practice, it should be easy to work out the middle step mentally.

Example 5.16

Is $4x^2 - 40xy + 25y^2$ a perfect square?

Solution

$4x^2 = (2x)^2$ and $25y^2 = (5y)^2$.

Thus, if the expression is a perfect square it must be the square of $2x - 5y$.

But $(2x - 5y)^2 = 4x^2 + 25y^2 - 20xy$.

$\therefore$ $4x^2 - 40xy + 25y^2$ *is not a perfect square.*

Note:

Instead of expanding $(2x - 5y)^2$ in order to compare the middle term with that of the given expression, we could alternatively first check if $-40xy$ is equal to $2(2x)(-5y)$. In this case, it is not. This indicates that $4x^2 - 40xy + 25y^2$ is not a perfect square.

Example 5.17

Complete $9p^2 - 12pq + 4q^2 = (\quad)^2$.

Solution

Factorising:

$$9p^2 - 12pq + 4q^2 = 9p^2 - 6pq - 6pq + 4q^2$$
$$= 3p(3p - 2q) - 2q(3p - 2q)$$
$$= (3p - 2q)(3p - 2q)$$
$$= (3p - 2q)^2$$

Alternative method

Since $9p^2 - 12pq + 4q^2$ is a perfect square, we realise that

$9p^2 = (3p)^2$, $4q^2 = (2q)^2$ *and*
$2 \times (3p) \times (2q) = 12pq$

$\therefore \quad 9p^2 - 12pq + 4q^2 = (3p - 2q)^2$

Notice that the second method is quicker and shorter. Thus, given an expression that is a perfect square, we need not use the grouping method when factorising it.

Exercise 5.5

1. Write down the squares of the following.
   (a) $x + 5$ (b) $x - 6$ (c) $3 - x$
   (d) $-y - 4$ (e) $2x + 9$ (f) $2y - 7$
   (g) $4x - y$ (h) $5y - 4z$ (i) $x + \frac{1}{x}$
2. State whether each of the following expressions is a perfect square. If it is, write it in the form $(a + b)^2$ or $(a - b)^2$.
   (a) $x^2 + 6x + 9$ (b) $x^2 - 9x + 9$
   (c) $x^2 - 8x + 16$ (d) $x^2 - 2x + 1$
   (e) $y^2 - 10y - 25$
   (f) $4 - 12m - 9m^2$
   (g) $a^2 + 3a + 6$ (h) $t^2 + 2t + 1$
   (i) $16a^2 - 20ab + 25b^2$
   (j) $25p^2 - 70pq + 49q^2$

5.2.3.2 Completing squares and solving quadratic equations

Activity 5.6

1. What must be done to make the following quadratic expressions perfect squares?
   (a) $x^2 + 4x$ (b) $x^2 + 9$
   (c) $4x^2 + 25$
2. Solve the following quadratic equations by completing squares method.
   (a) $x^2 + 6x + 4 = 0$
   (b) $4x^2 - 8x + 10 = 0$

Completing the square when the coefficient of x^2 is one

The quantity to be added is the **square of half of the coefficient of *x*** (or whatever letter is involved), i.e. to make $x^2 + bx$ a perfect square, we add $\left(\frac{b}{2}\right)^2$ to get

$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$, where b is any positive or negative number.

In some cases, it is not possible to solve a quadratic equation by the method of factorisation because the LHS does not factorise. In such a case, we first rearrange the equation to make the LHS a perfect square, i.e. we **complete** the square on the LHS.

If LHS of an equation factorises, it is better to use the method of factorisation instead of completing the square.

Example 5.18

What must be added to $x^2 + 8x$ to make the result a perfect square?

Solution

Suppose $x^2 + 8x + k$ is a perfect square and that it is equal to $(x + a)^2$, i.e.

$$x^2 + 8x + k = (x + a)^2.$$

We know that

$(x + a)^2 = x^2 + 2ax + a^2$

$\therefore \; x^2 + 8x + k = x^2 + 2ax + a^2.$

The expression on the LHS and that on the RHS can only be equal if the corresponding terms are equal.

Comparing coefficients of x:

$$2a = 8$$

$$\therefore \quad a = 4.$$

Comparing the constant terms:

$$k = a^2 = 4^2 = 16.$$

$\therefore$ *16 must be added to the expression* $x^2 + 8x$.

Then $x^2 + 8x + 16 = (x + 4)^2$.

Check by opening the bracket on the RHS!

Example 5.19

Find the term that must be added to $p^2 - 6p$ *to make the expression a perfect square.*

Solution

Suppose $p^2 - 6p + k$ *is a perfect square.*

Then, $p^2 - 6p + k = (p - a)^2$

i.e. $p^2 - 6p + k = p^2 - 2ap + a^2$

Comparing coefficients of p:

$$-2a = -6$$

$$\therefore \; a = 3.$$

Comparing the constant terms:

$$k = a^2 = 3^2 = 9$$

$\therefore$ *9 must be added to the expression.*

Then, $p^2 - 6p + 9 = (p - 3)^2$.

Note:

In Example 5.18, the coefficient of x is 8; half of 8 is 4; and the square of 4 is 16. Hence, 16 must be added.

In Example 5.19, the coefficient of p is –6; half of –6 is –3; and the square of –3 is 9. Hence, 9 must be added.

Example 5.20

Solve the equation $x^2 + 8x + 9 = 0$.

Solution

The LHS does not factorise and so we use the method of completing the square.

$$x^2 + 8x + 9 = 0$$

$$x^2 + 8x = -9 \quad \text{(subtracting 9 from both sides)}$$

$$x^2 + 8x + \left(\frac{8}{2}\right)^2 = -9 + \left(\frac{8}{2}\right)^2 \quad \left(\text{adding } \left(\frac{8}{2}\right)^2 \text{ to both sides}\right)$$

$$x^2 + 8x + 4^2 = 7$$

$$\Rightarrow \quad (x + 4)^2 = 7$$

$$\Rightarrow \quad x + 4 = \pm\sqrt{7}$$

$$\text{i.e.} \quad x = -4 \pm \sqrt{7}.$$

$$\text{Thus,} \quad x = {}^-4 + \sqrt{7} \text{ or } -4 - \sqrt{7}.$$

Example 5.21

Solve the equation $q^2 - 5q - 2 = 0$.

Solution

The LHS does not factorise, so we use the method of completing the square.

$$q^2 - 5q - 2 = 0$$

$$q^2 - 5q = 2$$

Add the square of $\left(\frac{-5}{2}\right)^2$ *to both sides*

$$q^2 - 5q + \left(\frac{5}{2}\right)^2 = 2 + \left(\frac{5}{2}\right)^2$$

$$= 2 + \frac{25}{4}$$

$$= \frac{8 + 25}{4}$$

$$\therefore \left(q - \frac{5}{2}\right)^2 = \frac{33}{4}$$

$$\Rightarrow \quad q - \frac{5}{2} = \pm\sqrt{\frac{33}{4}}$$

$$\Rightarrow q = \frac{5}{2} \pm \sqrt{\frac{33}{4}} = \pm\sqrt{\frac{33}{4}}$$

$$= \frac{5 \pm \sqrt{33}}{2}$$

$$= \frac{5 + \sqrt{33}}{2} \text{ or } \frac{5 - \sqrt{33}}{2}$$

Exercise 5.6

1. What term must be added to each of the expressions below to make the expression a perfect square? Write each expression as $(a + b)^2$ or $(a - b)^2$.

   (a) $y^2 + 10y$ (b) $x^2 - 8x$
   (c) $p^2 + 6pq$ (d) $d^2 + 5d$
   (e) $a^2 - 12ab$ (f) $q^2 - 7q$
   (g) $n^2 + n$ (h) $m^2 - mn$

2. Solve the equations below by factorising where possible, otherwise by completing the square. Do not put the answer in decimal form.

   (a) $x^2 - 4x - 21 = 0$
   (b) $y^2 + 3y - 10 = 0$
   (c) $x^2 + 3x - 11 = 0$
   (d) $d^2 + 4d - 4 = 0$
   (e) $p^2 + 3p - 2 = 0$
   (f) $25 - 10x + x^2 = 0$
   (g) $n^2 - 14n + 2 = 0$
   (h) $t^2 - 15t - 4 = 0$

Completing the square when the coefficient of x^2 is not equal to one

If the quadratic equation is of the form $ax^2 + bx + c = 0$, where $a \neq 1$ and the LHS does not factorise, first divide both sides by a to make the coefficient of x^2 one and then complete the square.

Example 5.22

Solve the equation $2x^2 + 14x + 9 = 0$, giving your answer correct to 3 d.p.

Solution

$2x^2 + 14x + 9 = 0$

First make the coefficient of x^2 one by dividing both sides by 2.

The equation becomes

$$x^2 + 7x + 4\tfrac{1}{2} = 0$$
$$\Rightarrow x^2 + 7x = -4\tfrac{1}{2}.$$

Completing the square gives

$$x^2 + 7x + \left(\frac{7}{2}\right)^2 = -4\frac{1}{2} + \left(\frac{7}{2}\right)^2$$
$$\Rightarrow \left(x + \frac{7}{2}\right)^2 = -4\frac{1}{2} + \frac{49}{4}$$
$$= -\frac{9}{2} + \frac{49}{4}$$
$$= \frac{-18 + 49}{4}$$
$$= \frac{31}{4}$$
$$\therefore\ x + \frac{7}{2} = \pm\sqrt{\frac{31}{4}}$$
$$\Rightarrow x = \frac{-7}{2} \pm \frac{\sqrt{31}}{2}$$
$$= \frac{-7 \pm \sqrt{31}}{2}$$
$$= \frac{-7 \pm 5.568}{2}$$
$$\frac{7 + 5.568}{2} \text{ or} = \frac{7 - 5.568}{2}$$
$$\text{Either } x = \frac{-1.432}{4} \text{ or } \frac{-12.568}{4}$$
$$= -0.716 \text{ or } -6.284$$

Exercise 5.7

1. Solve the following quadratic equations by completing squares method.

   (a) $x^2 - 2x - 8 = 0$
   (b) $x^2 - 5x + 2 = 0$
   (c) $x^2 + 2x - 8 = 0$
   (d) $x^2 + 2x - 1 = 0$
   (e) $2x^2 - 10x = 0$
   (f) $2x^2 + 6x + 1 = 0$

2. Complete the squares in the following expressions.

   (a) $x^2 - 8x - 7 = 0$
   (b) $2x^2 + x - 6 = 0$

(c) $2x^2 - 5x + 2 = 0$

(d) $x^2 + 4x = 0$

(e) $3x^2 - 2x - 6 = 0$

(f) $x^2 + \frac{1}{2}x = 0$

3. Solve the following equations by factorisation if possible, otherwise by completing the square. Give your answers to 2 d.p. where necessary.

(a) $2x^2 - 3x = 9$

(b) $2y^2 - 4y + 1 = 0$

(c) $2b^2 + b + 1 = 0$

(d) $4x^2 - 8x = -1$

(e) $4p^2 - 8p + 3 = 0$

(f) $4p^2 = 8p + 3$

(g) $3e^2 = 9e - 2$

(h) $3a^2 - 2 = 12a$

(i) $5x^2 = -1 - 15x$

(j) $2z^2 + 10z + 5 = 0$

5.2.4 Solving quadratic equations by using formula method

Activity 5.7

1. Use the steps that are applied in solving quadratic equations by completing squares to obtain the values of x in terms of a, b and c for which $px^2 + qx + r = 0$
   (a) First divide the whole equation by p to make the coefficient of x^2 be one.
   (b) Solve for the value of x by completing the square.
   (c) Make x the subject of the formula.
2. Use the expression you obtained in step 1 to solve the equation $2x^2 - 3x - 4 = 0$ by substituting $p = 2$, $q = -3$ and $r = -4$.
3. Solve the quadratic equation $2x^2 - 5x + 3 = 0$ by factorization method.
4. Compare your results for step 2 and step 3. What do you notice?

Consider the quadratic equation $ax^2 + bx + c = 0$. Let us solve by completing square method.

1. Make sure the coefficient of x^2 is one. This is done by dividing by a throughout the given equation.

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

2. Half the coefficient of x, square it and subtract it from the given expression.

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

3. Take the constant part to the right hand side and find the square root both sides and solve for the values of x.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{c}{\text{a}} - \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

$$x = \frac{-b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

$$= \frac{-b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

Simplifying the expression under the square root using LCM and taking square root of denominator we get

$$x = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is called the **quadratic formula** and it is used to solve any quadratic equation provided the coefficients a, b and c are all known.

Example 5.23

Solve the equation $2x^2 - 3x - 4 = 0$ with the aid of a quadratic formula.

Solution

In this case $a = 2, b = -3, c = -4$.

$$x = \frac{-(-3)^2 \pm \sqrt{(-3)^3 - (4 \times 2 \times -4)}}{2 \times 2}$$

$$= \frac{3 \pm \sqrt{(9 + 32)}}{4} = \frac{3 \pm \sqrt{(41)}}{4} = \frac{3 \pm \sqrt{41}}{4}$$

$$x = \frac{3 \pm \sqrt{41}}{4}$$

$$x = \frac{3 \pm 6.403}{4}$$

$$x = \frac{3 + 6.403}{4} \text{ or } x = \frac{3 - 6.403}{4}$$

$$x = 2.351 \text{ or } -0.851$$

Example 5.24

Solve the equation $2x^2 + 7x = 2$.

Solution

Writing the equation in the general form $ax^2 + bx + c = 0$ we get

$2x^2 + 7x - 2 = 0.$

Hence $a= 2, b = 7, c = -2$

Substitute these values in

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{49 - (4 \times 2 \times -2)}}{2 \times 2} = \frac{-7 \pm \sqrt{65}}{4}$$

Hence $x = 0.27$ or $x = -3.77$ (2 d.p).

Exercise 5.8

1. Solve the following with the aid of quadratic formula, giving answers to two decimal places where necessary:
   - (a) $2x^2 + 11x + 5 = 0$
   - (b) $3x^2 + 11x + 6 = 0$
   - (c) $6x^2 + 7x + 2=0$
   - (d) $3x^2 - 10x + 3 = 0$
   - (e) $5x^2 - 7x + 2 = 0$
   - (f) $6x^2 - 11x + 3 = 0$
   - (g) $2x^2 + 6x + 3 = 0$
   - (h) $x^2 + 4x + 1 = 0$
2. From the questions below, circle the correct solution.
   - (a) When the quadratic equation $2x^2 + 5x - 3 = 0$ is solved, the results obtained are;
     - (i) -3 and 2
     - (ii) -3 and $\frac{1}{2}$
     - (iii) 4 and -2
     - (iv) 6 and -1
   - (b) The equation $2x^2 - x - 6 = 0$ when solved gives the values of x as;
     - (i) 2 and -3
     - (ii) -1 and 3
     - (iii) 3 and -2
     - (iv) None of the above
   - (c) Which of the solutions below is suitable solution for $2x^2 + 3x - 5 = 0$?
     - (i) -2.5 and 1
     - (ii) -1.5 and 3
     - (iii) -3.4 and -2
     - (iv) None of the above
3. Write true/false for the statements below.
   - (a) 1 and 9 are the solutions to $x^2 - 10x + 9 = 0$
   - (b) $3x^2 - 10x + 3 = 0$ is a perfect square
   - (c) $x^2 + 9x + 14 = 0$ has no solution in the field of real numbers
   - (d) $x^2 + 5x - 500 = 0$ has two solutions one is positive and another is negative
   - (e) The equation $x^2 - x - 12 = 0$ cannot be solved by factorisation method

5.2.5 Solving Quadratic equations by synthetic division method

Activity 5.8

Use internet or library to research about factorisation of quadratic equations and polynomials by using Horner's rule.

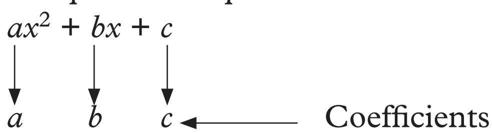
List the steps that are followed using your own words.

Consider the quadratic equation $ax^2 + bx + c = 0$.

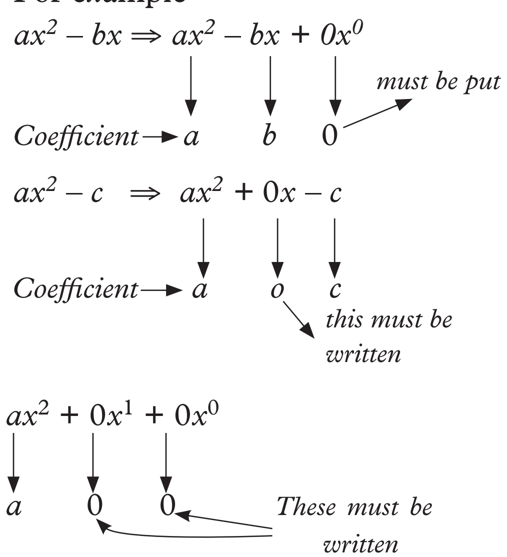
To solve this quadratic equation, we proceed as follows:

Steps

1. Write the coefficients of all terms including the constant term in descending powers of the variable in the quadratic equation.



2. In case there are missing powers of the variable in the equation, you should represent them with zero coefficients.
   For example



3. Identify all the factors of the constant c both positive and negative. Let the factors of c be e, f, g, h etc.

   Choose the factor of c which can reduce the quadratic expression to zero. Let the factor of c be f. This means that f is one of the solutions of the quadratic equation provided. i.e. $x = f$ and the linear factor is $(x - f)$

4. Arrange the coefficients and the factor f as shown below.

f	a	b	c

5. Proceed to get the other factor of the quadratic equation as follows.

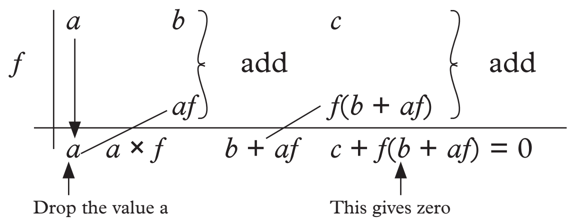


Fig. 5.7

For vertical patterns, add terms while for the diagonal patterns, multiply by factor f.

Hence on dividing the expression

$ax^2 + bx + c$ by $(x - f)$, we should get the other factor (quotient) as $ax + (b + af)$.

Thus the quadratic equation $ax^2 + bx + c = (x - f)(ax + (b + af))$ in factorised form.

We therefore solve the values of x from the factors $(x - f)(ax + (b + af)) = 0$, meaning $x - f = 0$ and $ax + (b + af) = 0$

Example 5.25

Solve the quadratic equation $x^2 - x - 6 = 0$ using synthetic division method.

Solution

The factors of 6 are ±1, ±2, ±3 and –6. The factor of -6 that can reduce the quadratic equation to zero is 3.

	1	–1	6
3	↓	3	6
	1	2	0

The values 1 and 2 obtained are the coefficients of the next linear factor called quotient.

Here we then get $(x-3)(x+2)=0$

Hence $x=3$ *and* $x=-2$ *are the roots of the given quadratic equation.*

Example 5.26

Solve the quadratic equation $3x^2-5x-2=0$ *by synthetic division method.*

Solution

The factor of –2 that can reduce the quadratic expression to zero is +2.

The coefficients of the quadratic equation are 3, –5 and –2.

3	–5	–2	
2	↓	6	2
	3	1	0

Then we have $(x-2)(3x+1)=0$

Hence $x=2$ *and* $x=\frac{1}{3}$.

Exercise 5.9

Solve the following quadratic equations by synthetic division method.

1. $x^2+x-12=0$
2. $x^2+9x+14=0$
3. $u^2+6u+9=0$
4. $a^2-2a+1=0$
5. $x^2+10x=24$
6. $v^2-36=0$
7. $x^2-11x-12=0$
8. $t^2-t-42=0$
9. $y^2+8y=0$
10. $x^2=4x-3$
11. $4x^2-9=0$

5.3 Problems involving quadratic equations

Activity 5.9

A 3 hour cruise ship goes 15 km upstream and then back again. The water in the river has a speed of 2 km/h. Let x represent the speed of the ship.

(a) Write down an expression for the speed of the ship upstream.

(b) Write down an expression for the speed of the ship downstream.

(c) Find expression for the time taken by the ship to move up and downstream.

(d) Write down and simplify an expression for the total time taken by the ship to move up and downstream.

(e) What name can be given to the expression obtained in (d) above?

(f) What is the ship's speed and how long was the upstream journey?

Quadratic equation can be modelled out of a well given data. There is no clear and specific method for this apart from reading and understanding the problem word by word and then applying what you have understood to model out a correct mathematical expression.

Example 5.27

The perimeter of a rectangle is 46 cm. If the diagonal is 17 cm, find the width of the rectangle.

Solution

Let the width of the rectangle be x cm.

Since the perimeter is 46 cm, the sum of the length and the width is 23 cm. Therefore, length of rectangle = (23 – x) cm.

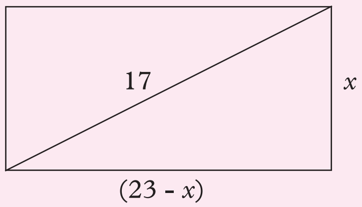


Fig. 5.1

By Pythagoras' theorem

$x^2 + (23 - x)^2 = 17^2$

$x^2 + 529 - 46x + x^2 = 289$

$2x^2 - 46x + 240 = 0$

$x^2 - 23x + 120 = 0$

$(x - 15)(x - 8) = 0$

$x = 15$ or $x = 8$

Note that the dimensions of the rectangle are 8 cm by 15 cm, whichever value of x is taken.

Therefore, the width of the rectangle is 8 cm

Example 5.28

A man bought a certain number of golf balls for 2,000 FRW. If each ball had cost 200 FRW less, he could have bought five more for the same money. How many golf balls did he buy?

Solution

Let the number of balls be x.

Cost of each ball is $\frac{2000}{x}$ *FRW*

If five more balls had been bought, cost of each ball now = $\frac{2000}{(x+5)}$

Therefore, $\frac{2000}{x} - \frac{2000}{(x+5)} = 200$

Multiplying by (x)

$x \times \frac{2\,000}{x} - x \times \frac{2\,000}{(x+5)} = 200x$

Multiplying by (x + 5)

$2000(x+5) - x \times \frac{2\,000}{(x+5)}(x+5) = 2000(x+5)$

$2\,000x + 10\,000 - 2\,000x$

$= 200x^2 + 1\,000x.$

$200x^2 + 1\,000x - 10\,000 = 0.$

$x^2 + 5x - 500 = 0$

$(x - 20)(x + 25) = 0.$

$x = 20$ or $x = -25.$

We discard x = –25 as meaningless.

The number of balls bought x = 20.

Exercise 5.10

Solve by forming a quadratic equation:

1. Two numbers which differ by 3, have a product of 88. Find the two numbers.
2. The product of two consecutive odd numbers is 143. Find those two odd numbers.
3. The length of a rectangle exceeds the width by 7 cm. If the area is 60 cm^2, find the length of the rectangle.
4. The length of a rectangle exceeds the width by 2 cm. If the diagonal is 10 cm long, find the width of the rectangle.
5. The area of the rectangle exceeds the area of the square by 24 m^2. Find x.

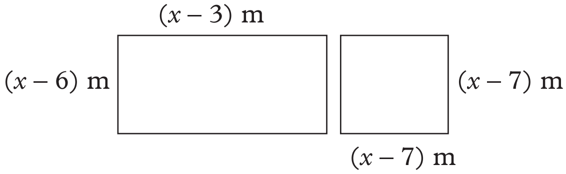


Fig. 5.2

6. The perimeter of a rectangle is 68 cm. If the diagonal is 26 cm, find the dimensions of the rectangle.

7. A man walks a certain distance due North and then the same distance plus a further 7 km due East. If the final distance from the starting point is 17 km, find the distances he walks north and East.
8. A farmer makes a profit of x FRW on each of the $(x + 5)$ eggs her hen lays. If her total profit was 84,000 FRW, find the number of eggs the hen lays.
9. A number exceeds four times its reciprocal by 3. Find the number.
10. Two numbers differ by 3. The sum of their reciprocals is $\frac{7}{10}$; find the numbers.

Unit Summary

- **A quadratic equation:** This is an equation that is of the form $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$.

  Examples of quadratic equations are $x^2 + 9x + 14 = 0$, $u^2 - 5u + 4 = 0$, $7 - 6r + r^2 = 0$ etc.
- Quadratic equations can be solved by the following methods.
  - By factorisation method
  - Graphical method
  - Completing squares method
  - Quadratic formula method
  - Synthetic division method
- In order to solve a quadratic equation, the quadratic expression is factorized so that the equation is in the form $(x + a)(x + b) = 0$.

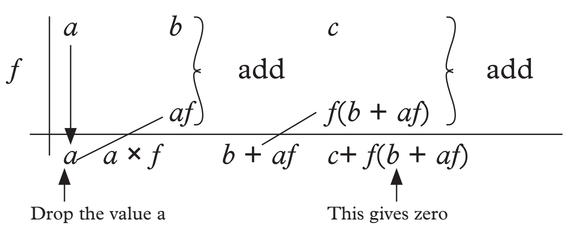
  Then either $(x + a) = 0$ or $(x + b) = 0$

  Thus $x = -a$ and $x = -b$
- In a quadratic function graph, the x-coordinate of the point where the graph cuts x-axis gives the solution to the quadratic equation represented by the function.
  (a) When the graph cuts the x-axis at one point, then the equation has one repeated solution.
  (b) When the graph cuts x-axis at two points, then the equation has two different solutions.
  (c) When the graph does not cut x-axis at any point, then the equation has no solution in the field of real numbers.
- Before solving any quadratic equation by completing squares method, it is better to understand what perfect squares are.

  For example $x^2 + 6x + 9 = (x + 3)^2$ is a perfect square.

  Remember that $(a + b)^2 = a^2 + 2ab + b^2$ and that $(a - b)^2 = a^2 - 2ab + b^2$
- The quadratic formula

  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is also suitable for solving quadratic equation $ax^2 + bx + c = 0$ provide the coefficients a, b and c are all known.
- The quadratic equation $ax^2 + bx + c = 0$ can also be solved by synthetic division method as long as the value $x = $ f is the factor of constant c. We follow the trend below.



For vertical patterns, add terms while for the diagonal patterns, multiply by factor f.

Hence, on dividing the expression $ax^2 + bx + c$ by $(x - f)$, we should get the other factor (quotient) as

$ax + (b + af)$

Thus the quadratic equation $ax^2 + bx + c = (x - f)(ax + (b + af))$ in factorised form.

We therefore solve the values of x from the factors $(x - f)(ax + (b + af)) = 0$. Meaning $x - f = 0$ and $ax + (b + af) = 0$

Unit 5 Test

1. A cyclist travels 40 km at a speed of x km/h. Find the time taken in terms of x. Find the time taken when his speed is reduced by 2 km/h. If the difference between the times is 1 hour, find the value of x.
2. A train normally travels 240 km at a certain speed. One day, due to bad weather, the train's speed is reduced by 20 km/h so that the journey takes two hours longer. Find the normal speed.
3. An aircraft flies a certain distance on a bearing of 135° and then twice the distance o a bearing of 225°. Its distance from the starting point is then 350 km. Find the length of the first part of the journey.
4. In the following figure, ABCD is a rectangle with AB = 12 cm and BC = 7 cm. AK = BL = CM = DN = x cm. If the area of KLMN is 54 cm^2 find x.

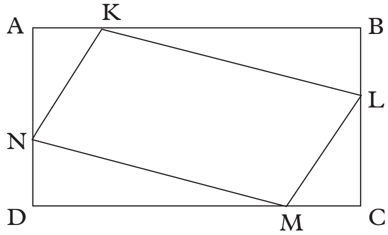


Fig. 5.3

5. The numerator of a fraction is 1 less than the denominator. When both numerator and denominator are increased by 2, the fraction is increased $\frac{1}{12}$. Find the original fraction.
6. The perimeters of a square and a rectangle are equal. One side of the rectangle is 11 cm and the area of the square is 4 cm^2 more than the area of the rectangle. Find the side of the square.
7. In a right angled triangle PQR, $\angle Q = 90°$, QR = x cm, PQ = $(2x - 2)$ cm and PR = 30 cm. Find x.
8. Draw the graph of $y = x^2 - 4x + 4$ for values of x from -1 to $+5$. Solve from your graph the equations:
   (a) $x^2 - 4x + 4 = 0$
   (b) $x^2 - 4x + 1 = 0$
   (c) $x^2 - 4x - 1 = 0$.
9. Solve the following quadratic equations by synthetic division method.
   (a) $x^2 + x - 12 = 0$
   (b) $x^2 + 9x + 14 = 0$.

6 LINEAR AND QUADRATIC FUNCTIONS

Key unit competence: By the end of this unit, learners should be able to solve problems involving linear and quadratic functions and interpret the graphs of quadratic functions.

Unit outline

- Linear functions
  - Definition of linear functions.
  - Slope/gradient of a linear function
  - Cartesian equation of line
  - Parallel and Perpendicular lines
- Quadratic functions
  - Table of Values
  - Vertex of a parabola and axis of symmetry.
  - Intercepts
  - Graph in Cartesian plane

Introduction

Unit Focus Activity

1. Consider line L_1 that passes through points A(3, 10) and B(6, 7) and L_2 that passes through point C(1, 2) and D(3, 8). Determine without drawing the line that is steeper.
2. The straight line $y = \frac{3}{2}x + 9$ meets the y-axis at A and the x-axis at B.
   (a) State the coordinates of A and B.
   (b) Find the equation of a line through A perpendicular to $y = \frac{3}{2}x + 9$.
   (c) Find the equation of a line through B that is parallel to $y = \frac{3}{2}x + 9$.
3. (a) Draw the graph of the quadratic function $y = 2x^2 + 5x - 9$ for the values of x from -4 to 2.
   Hence, solve the equation $2x^2 + 5x - 9 = 0$.
   (b) State the equation of the line of symmetry for the curve.
   (c) Use your graph to solve the equations $2x^2 + 3x - 4 = 0$

6.1 Linear functions

6.1.1 Definition of linear functions

Activity 6.1

1. Copy and complete the tables below.

x	-3	-2	-1	0	1	2	3
$y = 2x - 1$							

Table. 6.1

x	-3	-2	-1	0	1	2	3
$y = x^2 - 1$							

Table. 6.2

2. Use the coordinates from each table to plot the graphs on separate Cartesian planes.
3. What is your conclusion about the shapes of the graphs above?

Any function of the form $y = mx + c$ where constants m and c given is a straight line graph when drawn in a Cartesian plane. Such functions are known as **linear functions**. The constants can be zero or other integers.

The highest degree of a linear function is one.

Examples of linear functions are $y = 2x$, $y = 5x + 4$, $3x + y = 4$, and so many others.

6.1.2 Slope/Gradient of a linear function

Activity 6.2

1. Observe the lines in Fig. 6.1 below.

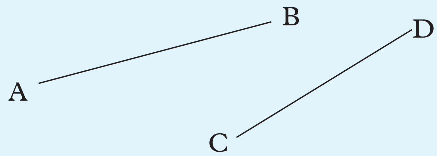


Fig. 6.1

Which line is steeper?

2. Draw the lines passing through the following points on the Cartesian plane:

   (a) A (2, 1) and B (6, 5)

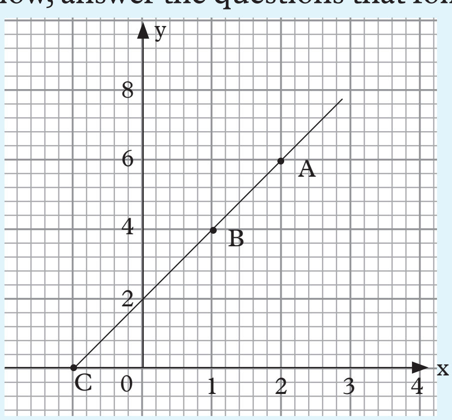
   (b) C (–2, 0) and D (1, 6)

   Which of the drawn lines in (2) above is steeper?

3. How could you have determined which line is steeper in step 2 without drawing?

Activity 6.3

From the graph on the Cartesian plane below, answer the questions that follow.



Graph 6.1

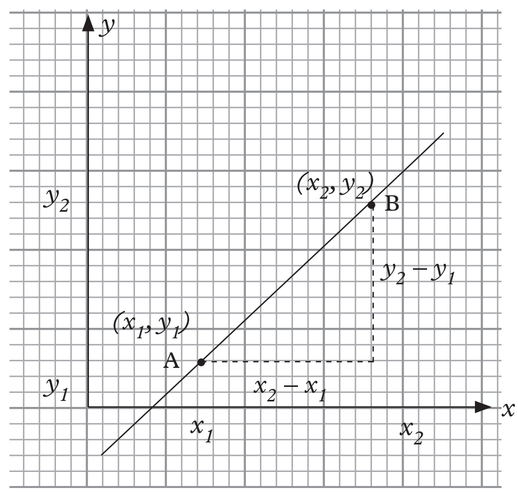
1. Read and record the coordinates of points A, B and C
2. Find gradient of AB and BC
3. What do you notice in part (2) above?

Every straight line has a slope with respect to the horizontal axis. The measure of the slope is called the gradient.

In the Cartesian plane, the gradient of a line is the measure of its slope or inclination to the x-axis. It is defined as the ratio of the change in y-coordinate (vertical) to the change in the x-coordinate (horizontal).

Thus, gradient/slope = $\frac{\textit{Change in y-coordinate}}{\textit{Change in x-coordinate}}$

Consider a line passing through the points A (x_1, y_1) and B(x_2, y_2).



Graph 6.2

From A to B, the change in the x-coordinate (horizontal change) is $x_2 - x_1$ and the change in the y-coordinate (vertical change) is $y_2 - y_1$.

By definition, gradient/slope

$$= \frac{\textit{Change in y-coordinate}}{\textit{Change in x-coordinate}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Or $= \frac{y_1 - y_2}{x_1 - x_2}$

Example 6.1

Find the gradient of a line passing through the points (2, 5) and (7, 9).

Solution

(2, 5) and (7, 9)

(x_1, y_1) and (x_2, y_2)

$$\text{Gradient} = \frac{\text{Change in } y\text{-coordinate}}{\text{Change in } x\text{-coordinate}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-5}{7-2} = \frac{4}{5}$$

$$\text{Or Gradient} = \frac{\text{Change in } y\text{-coordinate}}{\text{Change in } x\text{-coordinate}} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{5-9}{2-7} = \frac{-4}{-5} = \frac{4}{5}$$

Example 6.2

Given that line AB passes through the points A(–3, –4) and B(–1, –1); while line CD passes through the points C(–3, 5) and D(8, 1).

Find the gradients of line segments:

(a) AB (b) CD

Solution

(a) A(–3, –4) and B(–1, –1)

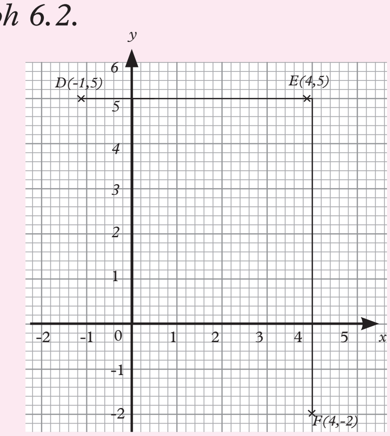
$$\text{Gradient } AB = \frac{\text{Change in } y\text{-coordinate}}{\text{Change in } x\text{-coordinate}} = \frac{-4-(-1)}{-3-(-1)} = \frac{-3}{-2} = \frac{3}{2}$$

(b) C(–3, 5) and D(8, 1).

$$\text{Gradient } CD = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} = \frac{1-5}{8-(-3)} = \frac{-4}{11}$$

Example 6.3

Find the gradients of lines DE and EF in graph 6.2.



Graph 6.3

Solution

Moving from D to E, the

change in x-coordinate is $4 - {}^{-}1 = 5$,

change in y-coordinates is $5 - 5 = 0$.

$\therefore$ gradient of line DE is $\frac{0}{5} = 0$

Moving from F to E, the

change in x-coordinate is $4 - 4 = 0$,

change in y-coordinate is $5 - -2 = 7$

$\therefore$ gradient of line EF is $\frac{7}{0}$

The value of the gradient is ***undefined***. (See note below.)

Thus the gradient of line EF is undefined

Note:

$\frac{6}{3} = 2$ means that 2 (quotient) multiplied by 3 (divisor) equals 6 (dividend).

Similarly, $\frac{0}{7} = 0$ *means that 0 (quotient) multiplied by 7 (divisor) equals 0 (dividend), which is true.*

But, $\frac{7}{0} = 0$ *would mean 7 (quotient multiplied by 0 (divisor) equals 7 (dividend), which is not true.*

Likewise, if x is any number, $\frac{x}{0} = 0$ *has no meaning.*

Thus, $\frac{0}{x} = 0$ *while* $\frac{x}{0}$ *is undefined.*

We notice that:

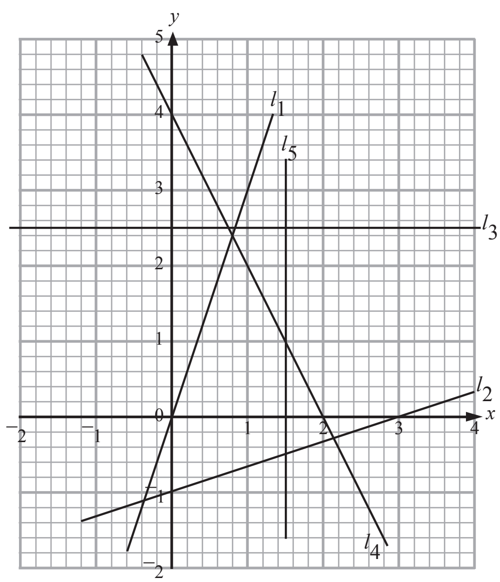
1. If, for an **increase** in the x-co-ordinate, there is **no change** in the y-co-ordinate, i.e. the line is **horizontal**, the gradient is **zero**.
2. If there is **no change** in the x-co-ordinate while there is an **increase** in the y-co-ordinate, i.e. the line is **vertical**, the gradient is **undefined**.

Note that any two points on a straight line gives the same value of the gradient/slope of that line.

Hence any two points on a line can be used to find its gradient.

Exercise 6.1

1. For each of the following pairs of points, find the change in the x-coordinate and the corresponding change in the y-coordinate. Hence find the gradients of the lines passing through them.
   (a) (0 , 2) and (3 , 4)
   (b) (0 , 2) and (5 , 0)
   (c) (–2 , –2) and (2 , 0)
   (d) (–1 , –2) and (1 , 8)
   (e) (–1 , 2) and (3 , –2)
   (f) (0 , –2) and (0 , 3)
   (g) (2 , –8) and (–2 , 8)
   (h) (–2 , 0) and (3 , 0)
2. Find the gradient of the line which passes through each of the following pairs of points.
   (a) (3 , 5) and (9 , 8)
   (b) (2 , 5) and (4 , 10)
   (c) (7 , 3) and (0 , 0)
   (d) (1 , 5) and (7 , 2)
   (e) (0 , 4) and (4 , 0)
   (f) (–2 , 3) and (5 , 5)
   (g) (–7 , 3) and (8 , –2)
   (h) (–3 , –4) and (3 , –4)
   (i) (–1, 4) and (–3, –1)
   (j) (3 , –1) and (3, 1)
3. In each of the following cases, the coordinates of a point and the gradient of a line through the point are given respectively. State the coordinates of two other points on the line.
   (a) (3 , 1), 3 (b) (4 , 5), 2
   (c) (–2 , 3), –1 (d) (5 , 5), – 1
   (e) (–4 , 3), undefined
   (f) (–4, 3), 0
4. Find the gradients of the lines l_1 to l_5 in graph 6.4.



Graph 6.4

DID YOU KNOW? Gradients are applied in Economics, Physics and Entrepreneurship. In Economics, we have demand and supply where the gradient of a demand curve is a negative indicating the relationship between demand and price, supply curve has positive gradient.

In Physics, gradients are applied in linear motion.

6.1.3 Cartesian Equation of a line

6.1.3.1 General form of Cartesian equation of a straight line

Activity 6.4

1. Write down the gradients and *y*-intercepts of the lines whose equations are:
   (a) $y = 3x + 4$ (b) $y = -2x + 5$
   (c) $y = 6 - 5x$ (d) $y = 7x$
2. Re-write the equations $x - 2y = -6$, $2x + 3y = 6$, $y = -1$
   $4x + 2y = 5$, $2y = x - 4$, and
   $4x - y = 6$ in form of $y = mx + c$.

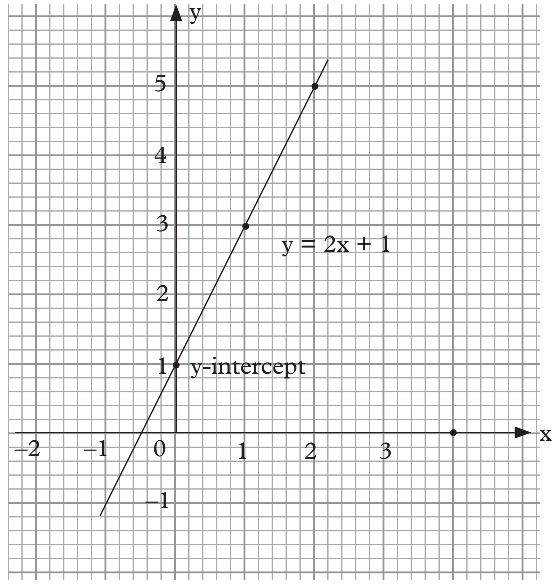
Copy and complete the table below.

Equation of the line	The form $y = mx + c$	m	c	gradi-ent	y-inter-cept
$x - 2y = -6$	$y = \frac{1}{2}x + 3$	$\frac{1}{2}$	3	$\frac{1}{2}$	3
$2x + 3y = 6$					
$4x + 2y = 5$					
$2y = x - 4$					
$y = -1$					
$4x - \frac{1}{2}y = 6$					

Table 6.3

At the beginning of this unit, we learnt that a linear function has a general form $y = mx + c$. In this case, m is the gradient of the line and c is the y – intercept.

The graph below shows the line $y = 2x + 1$.



Graph 6.5

The line has gradient of 2 and y-intercept of 1.

Example 6.4

Find the gradient of the line $3x - y = 2$ and draw the line on squared paper.

Solution

To find the gradient of the line, we need to first find any two points on the line.

We write the equation $3x - y = 2$ as $y = 3x - 2$, choose any two convenient values of x and find corresponding values of y.

For example, when

$x = 0, y = 3 \times 0 - 2 = -2,$

$\therefore$ *point* $(0, -2)$ *lies on the line.*

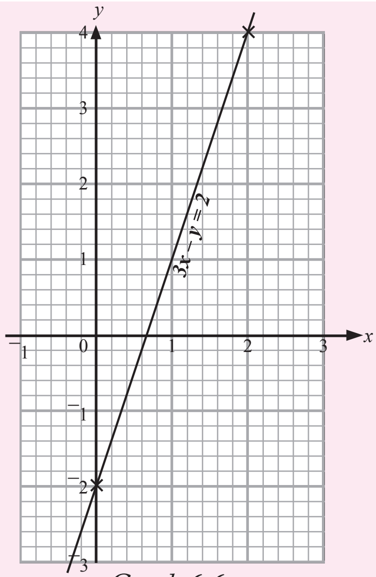
When $x = 2,\ y = 3 \times 2 - 2 = 4,$

$\therefore$ *point* $(2,4)$ *lies on the line.*

Thus gradient of the line is

$$\frac{4 - -2}{2 - 0} = \frac{4 + 2}{2 - 0} = \frac{6}{2} = 3$$

Fig. 6.6 shows the line $3x - y = 2$.



Graph 6.6

At what value does the line 3x – y = 2 cut the y-axis?
This value is referred to as the y-intercept. It is –2 for this example.

Example 6.5

Find the gradient and y-intercept of the line whose equation is 4x – 3y – 9 = 0. Sketch the line.

Solution

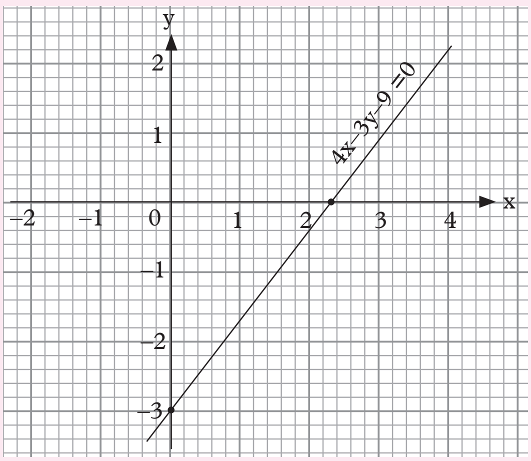
$4x - 3y - 9 = 0$ is equivalent to $y = \frac{4}{3}x - 3$.

Comparison with $y = mx + c$ gives

gradient, $m = \frac{4}{3}$,

y-intercept, $c = -3$.

Graph 6.7 is a sketch of the line.

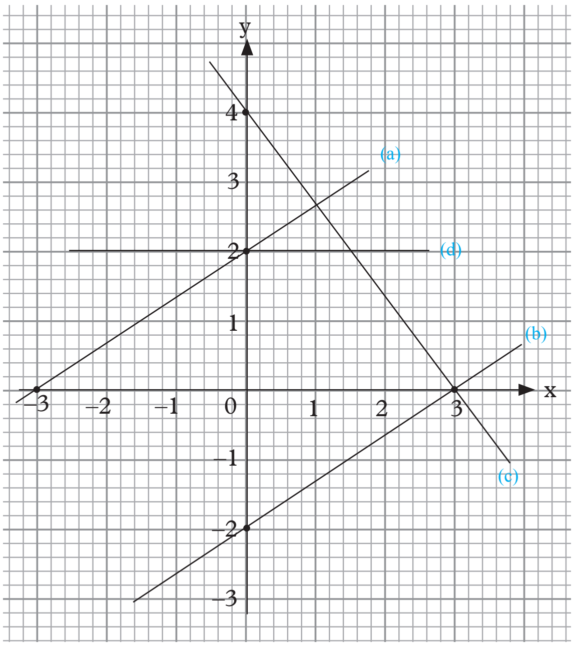


Graph 6.7

Note that for an increase of 3 units in the x-coordinate, the increase in the y-coordinate is 4 units. Hence, point (3, 1) is on the line.

Exercise 6.2

1. In each of the following cases, determine the gradient and y-intercept by writing the equation in the form $y = mx + c$. Sketch the line.
   (a) $5x = 2y$
   (b) $y = 2x + 1$
   (c) $2x + y = 3$
   (d) $4x - 2y + 3 = 0$
   (e) $2x + 3y = 3$
   (f) $5 = 5x - 2y$
   (g) $2x + 3y = 6$
   (h) $8 - 7x - 4y = 0$
2. Find the y-intercepts of the lines with the given gradients and passing through the given points.
   (a) 3, (2 , 6)
   (b) $\frac{5}{3}$, (–2 , 3)
   (c) –2, (7 , 4)
   (d) $-\frac{1}{4}$, (2 , 4)
   (e) 0, (–3 , –2)
   (f) Undefined, (1, 3)
3. Write down, in the gradient-intercept form, the equations of lines (a), (b), (c) and (d) in Graph 6.8.



Graph 6.8

6.2.3.2 Finding the equation of a straight line given gradient and a point on the line

Activity 6.5

You are given the straight line that passes through the point (3, -1) and its gradient is 2.

1. Taking (x) as any other general point through the line, form an expression for the gradient using the two points $(2, –1)$ and (x, y).
2. Find the equation of the line by equating an expression you obtained in step 1 to the gradient of 2. Simplify the equation in form $y = mx + c$

Example 6.6

A straight line with gradient 3 passes through the point A(3,–4). Find the equation of the line.

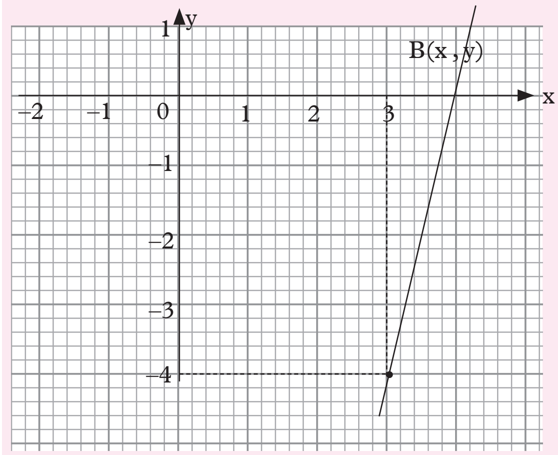
Solution

Fig. 6.9 is a sketch of the line.

The point B(x , y) represents any general point on the line.

Using points A and B, we have,

$$\text{gradient} = \frac{y--4}{x-3} = \frac{y+4}{x-3}$$



Graph 6.9

Since gradient $= 3,$

Then, $\frac{y+4}{x-3} = 3$

$\Leftrightarrow \; 3(x-3) = y + 4$

$\Leftrightarrow \; 3x - 9 = y + 4$

$\Leftrightarrow \; 3x - y = 13$ *(This is the equation of the line).*

In general, the equation of a straight line, of gradient m, which passes through a point (a, b) is given by

$$\frac{y-b}{x-a} = m.$$

Exercise 6.3

1. In each of the following cases, the gradient of a line and a point on the line are given. Find the equation of the line.

   (a) 3; (3, 1) (b) $\frac{1}{2}$; (4, 5)

   (c) –1; (–2, 3) (d) $-\frac{1}{5}$; (5, 5)

(e) 0; (–4, 3)
(f) Undefined; (–4, 3)
(g) $\frac{2}{3}$; (–3 , 0) (h) $-\frac{2}{3}$; (0 , 2)

2. In each of the following, find the equation of a line whose gradient and a point through which it passes are:
(a) –3; (0, 0) (b) $\frac{2}{3}$; $(\frac{1}{2}, \frac{1}{3})$
(c) $-\frac{5}{2}$; (4, 0) (d) 0; (4, 3)

3. Find the equations of lines described below.
(a) A line whose gradient is –2 and x-intercept is 1
(b) A line whose gradient is $-\frac{1}{5}$ and y-intercept is 4

4. The gradients of two lines l_1 and l_2 are $-\frac{1}{2}$ and 3 respectively. Find their equations if they meet at the point (2, 3).

5. Line L_1 has a gradient of -1 and passes through the point (3, 0). Line L_2 has a gradient of $\frac{2}{3}$ and passes through the point (4, 4). Draw the two lines on the same pair of axes and state their point of intersection.

6.1.2.3 Equation of a straight line given two points

Activity 6.6

You are provided with the points A(–1, 1) and B(3, 2) along the straight line.

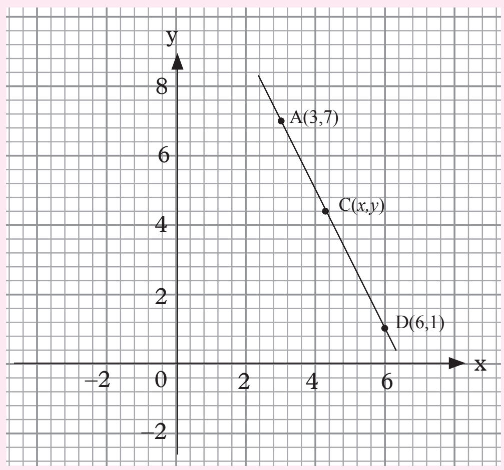
1. Find the gradient of the line joining the two points A and B.
2. Taking the point C (x, y) as the general point on the line and the point A(–1, 1), find an expression for the gradient of line AC.
3. Equate gradient obtained in step 1 to the gradient obtained in step 3 Simplify the result.
4. Repeat step (2) and (3) using (3, 2) and (x,y)
5. Compare the results. What do you notice?

Example 6.7

Find the equation of the straight line which passes through points A(3 , 7) and B(6 , 1).

Solution

Graph 6.10 is a sketch of the line.



Graph 6.10

Using points A and B, we have

Gradient $= \frac{1-7}{6-3} = \frac{-6}{3} = -2$

Point C(x, y) is any general point on line AB.

Using points A and C, we get

gradient $= \frac{y-7}{x-3}$

(We could equally well use points B and C. Since ACB is a straight line, the two values of the gradient are equal.)

$\therefore \quad \frac{y-7}{x-3} = -2$

$\Leftrightarrow \quad -2(x-3) = y-7$

$\Leftrightarrow \quad -2x+6 = y-7$

$\Leftrightarrow \quad 2x+y = 13$ *(This is the equation of the line)*

In general,
The equation of a straight line which passes through points (a, b) and (c, d) is given by,

$$\frac{y-b}{x-a} = \frac{d-b}{c-a} \text{ or } \frac{y-b}{x-a} = \frac{b-d}{a-c}$$

Example 6.8

Find the equation of the line passing through the points (4, 5) and (8, 7).

Solution

Gradient $= \frac{7-5}{8-4} = \frac{2}{4} = \frac{1}{2}$.

Let (x, y) be a point on the same line. Using one of the points, say (4, 5),

Gradient $= \frac{y-5}{x-4}$ *and the two gradients are equal.*

Therefore, $\frac{y-5}{x-4} = \frac{1}{2}$

$$y - 5 = \frac{1}{2}(x-4)$$

$$y - 5 = \frac{1}{2}x - 2$$

$$y = \frac{1}{2}x + 3$$

Example 6.9

Find the equation of a line with x-intercept –4 and y-intercept is 3.

Solution

At the x-intercept, $y = 0$ so we have (–4, 0).

At the y-intercept, $x = 0$ so we have (0, 3)

We therefore find the equation of the line by joining the points (–4, 0) and (0, 3)

Gradient $= \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$

Gradient $\frac{0-3}{-4-0} = \frac{-3}{-4} = \frac{3}{4}$

Taking a general point and the point

Gradient $\frac{y-3}{x-0}$

Equating the two gradients, we get

$$\frac{3}{4} = \frac{y-3}{x-0}$$

So, $3x = 4y - 12$ *we get* $y = \frac{3}{4}x + 3$

Exercise 6.4

1. Find the equation of the line that passes through the points
   (a) (–1, 1), (3, 2)
   (b) (7, 2) (4, 3)
   (c) (2, 5), (0, 5)
   (d) (5, –2), (6, 2)
   (e) (6, 3), (–6 , 2)
   (f) (2, –5), (2, 3)
   (g) $(\frac{1}{4}, \frac{1}{3})$, $(\frac{1}{3}, \frac{1}{4})$
   (h) (0.5, 0.3), (–0.2, –0.7)
2. (a) Find the equation of the straight line which passes through the points (0, 7) and (7, 0).
   (b) Show that the equation of the straight line which passes through $(0, a)$ and $(a, 0)$ is $x + y = a$.
3. A triangle has vertices A (–2, 0), B(–1,3) and C(2,3). Find the equations of the sides of the triangle.
4. Two lines, l_1 and l_2, both pass through the point $(4, k)$.
   (a) If l_1 passes through the point (5 , –3), and has a gradient $-1\frac{1}{3}$, find the value of k
   (b) If l_2 passes through (–14 , 0), find its equation
5. Find the equations of lines described below
   (a) A line whose x-intercept is 0 and passing through the point (–3,1)
   (b) A line whose y-intercept is –5 and passing through point (–4, –1)

(c) A line whose x and y- intercepts are 4 and –5

6. Line L_1 passes through the point (–1, 3) and has a gradient of 3. Line L_2 passes through the point (2, 3) and meets line L_1 at the point (0, 6).
   (a) Find the equations of the two lines.
   (b) Draw the lines L_1 and L_2 on the same pair of axes.

7. Determine the gradients and the y-intercepts of the straight lines:
   (a) $y = 8x + 1$
   (b) $y = x$
   (c) $y = 3 - 2x$
   (d) $y + x = 0$
   (e) $3y + x = 9$
   (f) $2x + 5y + 10 = 0$
   (g) $\frac{1}{2}y + \frac{1}{3}x = 2$
   (h) $\frac{2}{5}y + \frac{1}{2}x + 5 = 0$

8. Show that the point (–1, –4) lies on the line $y = 3x - 1$.

9. Show that the equation of the straight line passing through $(0, k)$ and $(k, 0)$ is $y + x = k$.

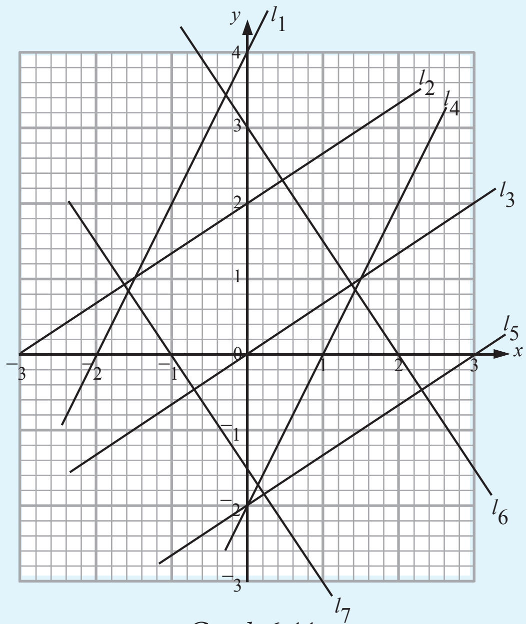
10. Given that the line $y = 3x + a$ passes through (1, 4), find the value of a.

6.1.4 Parallel and Perpendicular lines

6.1.4.1 Parallel lines

Activity 6.7

Consider Graph 6.11 below



Graph 6.11

1. Which sets of lines are parallel?
2. Calculate the gradients of all the lines. What do you notice about gradient of parallel lines?

Lines are parallel if they have the same gradient.

Consider two lines $y = m_1x + c_1$ and $y = m_2x + c_2$. These lines are parallel only and only if $m_1 = m_2$

Example 6.10

Find the equation of a line which passes through the point (3, 5) and is parallel to $2y = 2 - 6x$.

Solution

The equation of the required line is in the form of $y = mx + c$. The gradient of the line $2y = -6x + 2 \Rightarrow y = -3x + 1$ is –3.

Since the two lines are parallel $m = -3$. Thus, $y = -3x + c$.

The fact that the required line passes through (3,5),

$5 = -3 \times 3 + c$

$5 = -9 + c$

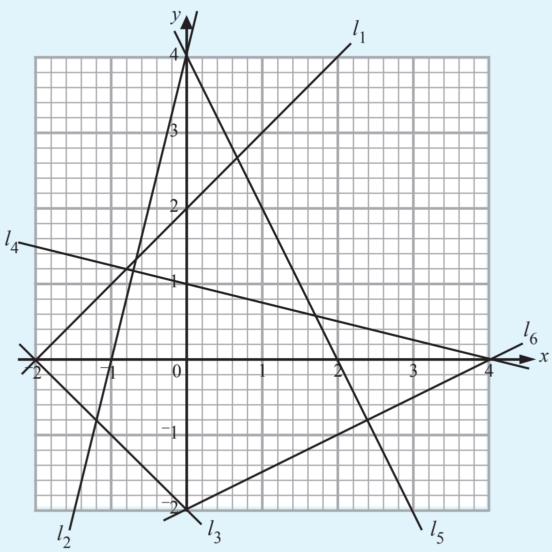
$c = 14$

The required equation is $y = -3x + 14$

6.1.4.2 Perpendicular lines

Activity 6.8

Consider graph 6.12 below.



Graph 6.12

1. Observe and state which pairs of lines are perpendicular?
2. (i) Calculate the gradients of all the lines, use the results to complete the table below.

Pair	Gradients of lines		Product of gradients ($m_1 \times m_2$)
	m_1	m_2	

Table. 6.4

(ii) What do you notice about $m_1 \times m_2$?

Lines are said to be perpendicular if the product of their gradients gives –1.

i.e. $m_1 \times m_2 = -1$

Example 6.11

Find the equation of the line through (5, 2) which is (a) parallel, (b) perpendicular to the line $5y - 2x = 10$.

Solution

The line $5y - 2x = 10$ *can be written as* $y = \frac{2}{5}x + 2.$

$\therefore$ *its gradient is* $\frac{2}{5}$.

(a) *The gradient of the line through (5, 2) is also* $\frac{2}{5}$ *since this line is parallel to* $5y - 2x = 10$.

If (x, y) is a general point on this line,

$$\text{gradient} = \frac{y-2}{x-5}$$

$$\therefore \frac{y-2}{x-5} = \frac{2}{5}$$

$$\Leftrightarrow 5y - 10 = 2x - 10$$

$$\Leftrightarrow 5y = 2x$$

$$\Leftrightarrow y = \frac{2}{5}x$$ *(This is the equation of the required line)*

(b) *The required line is perpendicular to* $5y - 2x = 10$. *If its gradient is m, then*

$$m \times \frac{2}{5} = -1$$

$$m = -\frac{5}{2}.$$

If (x, y) is a general point on the required line,

$$\text{then, gradient} = \frac{y-2}{x-5}$$

$$\therefore \frac{y-2}{x-5} = \frac{-5}{2}$$

$$\Leftrightarrow 2y - 4 = -5x + 25$$

$$\Leftrightarrow 2y = -5x + 29$$

$$\Leftrightarrow y = -\frac{5}{2}x + \frac{29}{2}$$ *(This is the equation of the required line).*

Exercise 6.5

1. Determine the gradients of the following pairs of equations and state whether their lines are parallel without drawing.

   (a) $y = 2x - 7$, $3y = 6x + 2$ (b) $y = 4$, $y = -3$

   (c) $y = 2x + 3$, $y = 4x + 6$

   (d) $5y + 3x + 1 = 0$, $10y + 6x - 1 = 0$

   (e) $2y + x = 2$, $3y + 2x = 0$ (f) $2x + y = 3$, $3x + y = 1$

   (g) $x + 2y = 4$, $x + 3y = 6$ (h) $y = 2x + 3$, $2y = 4x - 7$

   (i) $3y = 5x + 7$, $6y = 10x - 3$ (j) $5y = x + 2$, $4y = x + 3$

2. Without drawing, determine which of the following pairs of lines are perpendicular.

   (a) $y = 2x + 5$ and $2y + x = 3$

   (b) $2x - y = 7$ and $x + y = 5$

   (c) $3y = 2x + 1$ and $2y + 3x - 5 = 0$

   (d) $7x - 2y = -2$ and $14y - 4x = -1$

   (e) $y = \frac{3}{4}x - 2$ and the line through (8, 10) and (2, 2).

   (f) A line through (2, 2) and (10, 8) and another through (5, 6) and (8, 2).

3. A line through the points (–2,4) and (3, 5) is parallel to the line passing through the points (a, 6) and (–4,1). Find a.

4. Line L is parallel to a line whose equation is $y = 4x - 7$ and passes through the point (1, –2). Find the equation of line L.

5. Find the equation of the line that is parallel to another line whose equation is $4y + 5x = 6$ and passes through the point (8, 5).

6. Find the equation of the line that is parallel to another line whose equation is $y = \frac{2}{5}x + 2$ and passes through the point (–2, –3).

7. Write down the equation of the line perpendicular to:

   (a) $3x + 4y - 1 = 0$ and passes through (1, 2),

   (b) $y = \frac{3}{4}x + \frac{3}{4}$ and passes through the origin,

   (c) $3x - 2y + 7 = 0$ and passes through (–1, 0),

   (d) $5y + x + 4 = 0$ and passes through (3, 5).

8. Find the equation of the line that is parallel to another line whose equation is $y = \frac{2}{5}x + 2$ and passes through the point (–2, –3).

6.2 Quadratic functions

Activity 6.9

Study the following functions and decide which of them are quadratic functions. Give reasons to support your answer.

(a) $y = 3x + 4$

(b) $y = 3x^2 - 9x - 3$

(c) $y = x^3 - 4x^2 + 10$

(d) $y = x^2$

(e) $y = \frac{2x}{x^2 - 4 - 8}$

The expression $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$, is called a **quadratic function** of x or a function of the second degree (highest power of x is two).

Examples of quadratic functions are

(a) $f(x) = x^2 - 9$ (b) $f(x) = 2 - 3x + x^2$

(c) $f(x) = 2x^2 - 3x - 4$ (d) $y = x^2 + 8$

6.2.1 Table of values

Activity 6.10

Given the quadratic function $y = x^2 - x - 6$. (Hint: Refer to Unit 5)

1. Copy and complete the table below.

x	-4	-3	-2	-1	0	1	2	3	4
x^2									
$-x$									
-6									
$y = x^2 - x - 6$									

Table 6.5

2. State the coordinates from the table in ordered pairs (x, y)

Table of values are used to determine the coordinates that are used to draw the graph of a quadratic function.

To get the table of values, we need to have the domain (values of an independent variable) and then the domain is replaced in a given quadratic function to find range (values of dependent variables). The values obtained are useful for plotting the graph of a quadratic function.

All quadratic function graphs are parabolic in nature.

Example 6.12

Draw the table of values of $y = x^2$ and $y = -x^2$ for values of x between –5 and +5. Plot the graphs.

Solution

Table of $y = x^2$

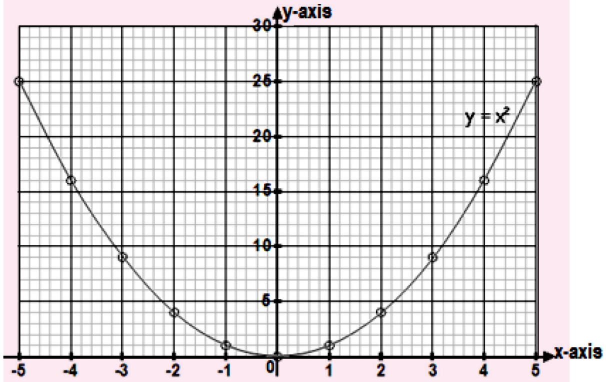
Make a table of values for (x, y)

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

Table 6.6

The values are:

(–5,25), (–4,16), (–3,9), (–2,4), (–1,1), (0,0), (1,1), (2,4), (3,9), (4,16), (5,25)



Graph 6.13

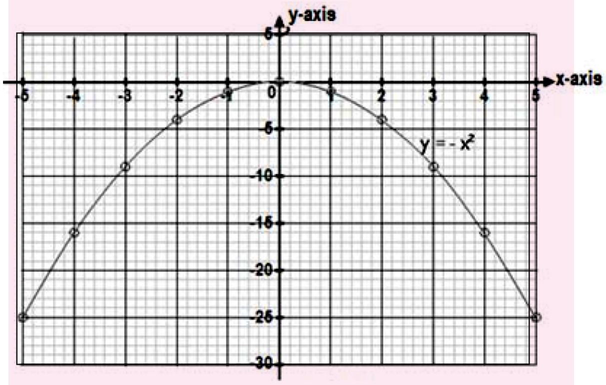
Table of $y = -x^2$

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	–25	–16	–9	–4	–1	0	–1	–4	–9	–16	–25

Table 6.7

The values are

(–5, –25), (–4, –16), (–3, –9), (–2, –4), (–1,–1), (0,0), (1, –1), (2, –4), (3, –9), (4,–16), (5, –25)



Graph 6.14

Example 6.13

Draw the table of $y = x^2 - 3x + 2$, for values of x between -1 and +4.

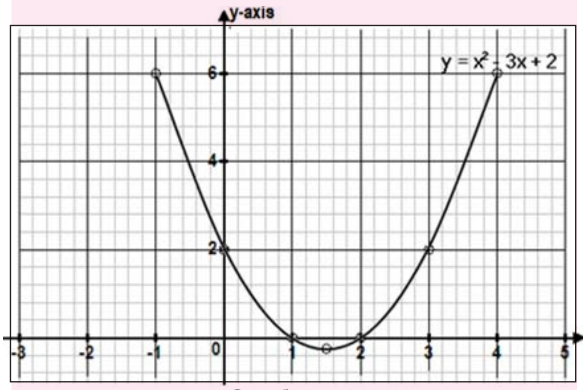
Solution

Make a table of values of x and y

x	–1	0	1	1.5	2	3	4
x^2	1	0	1	2.25	4	9	16
$-3x$	3	0	–3	–4.5	–6	–9	–12
+2	+2	+2	+2	+2	+2	+2	+2
	6	2	0	–0.25	0	2	6

Table 6.8

The values are (–1,6), (0,2), (1,0), (1.5,–0.25), (2,0), (3,2), (4,6)



Graph 6.15

Exercise 6.6

1. (a) Draw the table of $y = 1 + x - 2x^2$, taking values of x in the domain $-3 < x < 3$. State the coordinates obtained.

   (b) Use the same domain to draw the table of $y = 2x - 5$. State the coordinates obtained.

2. Draw the graph of $y = 2x^2 + x - 2$ from $x = -3$ to $x = 2$. Hence, find the appropriate values of the roots of the equation $2x^2 + x - 2 = 0$.

3. Copy and complete the following table of values for $y = 6 + 3x - 2x^2$. Plot the graph of the function.

x	-2	-1	0	0.5	1	2	3
6							
$3x$							
$-2x^2$							
y	-8	--	--	--	7	4	--

Table 6.9

4. Given that $y = x^2 - x - 2$ complete the following table for values of x and y. State the coordinates in ordered pairs (x, y) . Plot the graph.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
x^2											
$-x$											
–6											
y	--	1.75	0	-1.25	--	--	--	-1.25	--	1.75	--

Table 6.10

5. Given that $y = (3x + 1)(2x - 5)$, copy and complete the following table for values of x and y. State the coordinates in ordered pairs and plot the graph.

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3	3.5	4
$(3x + 1)$											
$(2x - 5)$											
y	14	3	-	-	-12	-11	-	-	-	23	-

Table 6.11

6.2.2 Determining the Vertex of a quadratic function and axis of symmetry from the graph.

Activity 6.11

Use internet or dictionary to explain the following terms.

(a) Line of symmetry

(b) Maximum point

(c) Minimum point

Any quadratic function has a graph which is symmetrical about a line which is parallel to the y-axis i.e. a line $x = h$ where h = constant value. This line is called **axis of symmetry** as shown in graph 6.16 below.

For any quadratic function $f(x) = ax^2 + bx + c$ whose axis of symmetry is the line $x = h$, the vertex is the point $(h, f(h))$.

Example 6.14

1. *Given that of $y = x^2 + 2x - 2$ is a quadratic function.*

(a) Prepare the table of values for the function $y = x^2 + 2x - 2$ for $-4 \le x \le 2$.

(b) Draw the graph.

(c) From the graph, identify;

(i) the value of y when x = 1.5

(ii) the value of x when $y = {}^-1$,

(iii) the least value of y.

(d) Determine the line of symetry of the function.

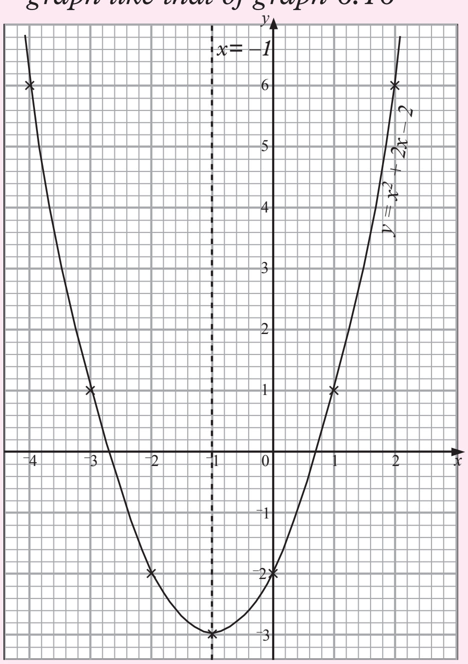
Solution

(a) Table below is the required table of values. Values of y are obtained by adding the values of x^2, 2x and $^-2$. Note that breaking down the expression into such components makes the working much easier.

x	–4	–3	–2	–1	0	1	2
x2	16	9	4	1	0	1	4
2x	–8	–6	–4	–2	0	2	4
–2	–2	–2	–2	–2	–2	–2	–2
y	6	1	–2	–3	–2	1	6

Table 6.12

(b) Choose a suitable scale and plot the values of y against the corresponding values of x. Join the various points with a continuous smooth curve to obtain a graph like that of graph 6.16



Graph 6.16

(c) From graph 6.16,

(i) when x = 1.5, y = 3,

(ii) when y = –1, x = –2.4 or 0.4,

(iii) the least value of y is –3.

(c) For the graph $y = x^2 + 2x - 2$, the symmetry is line x = –1.

Reason: It is the x-value of the lowest point on the graph.

Example 6.15

Draw the graph of $y = 2 + 2x - x^2$ for values of x from –2 to 4. From the graph, find:

(a) the maximum value of $2 + 2x - x^2$

(b) the value of x for which y is greatest

(c) the range of values of x for which y is positive

(d) the axis of symmetry.

Solution

The table 6.13 is the required table of values.

x	–2	–1	0	1	2	3	4
2	2	2	2	2	2	2	2
2x	–4	–2	0	2	4	6	8
–x2	–4	–1	0	–1	–4	–9	–16
y	–6	–1	2	3	2	–1	–6

Table 6.13

Fig. 6.17 shows the graph of $y = 2 + 2x - x^2$.

From the graph

(a) the maximum value of $2 + 2x - x^2$ is 3,

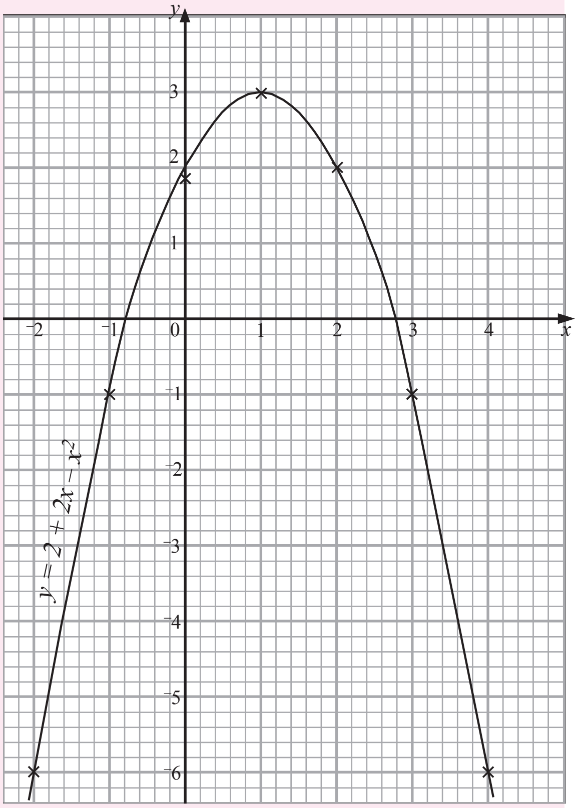
(b) the value of x for which y is greatest is 1,

(c) y is positive for all parts of the curve above the x-axis, i.e. when x > –0.8 and x < 2.8. In short, y is positive over the range $^-0.8 < x < 2.7$.

(d) the axis of symmetry for the graph of $y = 2 + 2x - x^2$ is $x = 1$.

Notice that the graph 6.17 is upside down as compared to graph 6.15.

This is always the case when the coefficient of x^2 is negative.



Graph 6.17

Exercise 6.7

1. Plot the graph of $y = x^2 - 4x + 5$ for $-1 < x < 5$. Use the graph to answer the questions below.
   (a) Where does the graph cut x-axis?
   (b) State the axis of symmetry of the graph.
   (c) Find the vertex of the graph.
   (d) Find the value of y when $x=3$.

2. Given that $y = x^2 - x - 2$ complete the following table for values of x and y.

x	x^2	$-x$	-2	y
– 2				
– 1				
0				
0.5				
1				
2				
3				

Table 6.14

   (a) State the co-ordinates in ordered pairs
   (b) Plot the graph of the function
   (c) Find the axis of symmetry from the graph plotted in (b).
   (d) Find the vertex of the function from the graph plotted in (b).

3. The table below shows the values of $y = x^2 - 3x + 2$

x	-1	0	1	2	3	4
y	6	2	0	0	2	6

Table 6.15

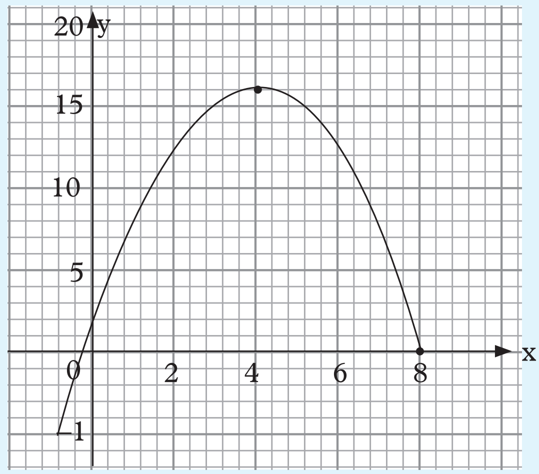
   (a) Using a suitable scale, draw a graph of $y = x^2 - 3x + 2$.
   (b) What is the axis of symmetry of $y = x^2 - 3x + 2$?
   (c) What is the vertex of the function $y = x^2 - 3x + 2$.
   (d) Use the graph drawn to solve $y = x^2 - 3x + 2$.

4. (a) Plot the graphs of $y = 3x - x^2$ and $y = x^2 - 3x$ in for $-1 \leq x \leq 4$ on the same axes.
   (b) What is the relationship between the two graphs plotted in 4(a) above?

6.2.3 Determining the intercepts, vertices and sketching quadratic functions

Activity 6.12

Consider the graph in Graph 6.18 below,



Graph 6.18

1. Read and record the vertex of the graph.
2. What is the axis of symmetry from the graph?
3. Read and record the points where the graph cuts and axes.

The vertex of a quadratic function is the point where the function crosses its axis of symmetry.

If the coefficient of the x^2 term is positive, the vertex will be the lowest point on the graph, the point at the bottom of the U-shape. If the coefficient of the term x^2 is negative, the vertex will be the highest point on the graph, the point at the top of the ∩-shape. The shapes are as below.

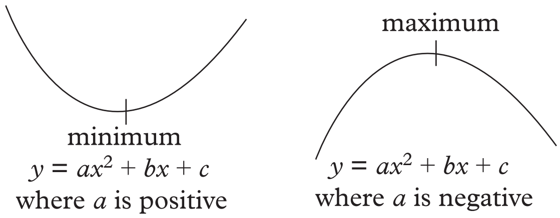


Fig. 6.2

The standard equation of a quadratic function is $y = ax^2 + bx + c$.

Since the quadratic expression written as $f(x) = ax^2 + bx + c$, then we can get the y-coordinate of the vertex by substituting the x-coordinate $= -\frac{b}{2a}$.

So the vertex becomes $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

The axis of symmetry is the x-coordinate of the quadratic function. Axis of symmetry is therefore calculated from $x = -\frac{b}{2a}$.

The intercepts with axes are the points where a quadratic function cuts the axes. There are two intercepts i.e. x-intercept and y-intercept. x-intercept for any quadratic expression is calculated by letting $y = 0$ and y intercept is calculated by letting $x = 0$

The graph of a quadratic expression can be sketched without table of values as long as the following are known.

(a) The vertex

(b) The x-intercepts

(c) The y-intercept

Example 6.16

Find the vertex of $y = 3x^2 + 12x - 12$. State the axis of symmetry.

Solution

The coefficients are $a = 3, b = 12$ and $c = -12$

The x-coordinate $h = -\frac{b}{2a}, = \frac{-12}{2(3)} = -2$.

Substituting the x-coordinate to get y coordinate, we have

$y = 3(-2)^2 + 12(-2) - 12 = -24.$

The vertex is at (–2, –24)

The axis of symmetry is the line $x = -2$.

Example 6.17

Find the vertex and axis of symmetry of the parabolic curve $y = 2x^2 - 8x + 6$

Solution

The coefficients are $a = 2$, $b = -8$ and $c = 6$

The x-coordinate of the vertex is

$h = -\frac{b}{2a} = -\frac{(-8)}{2(2)} = \frac{8}{4} = 2.$

The y-coordinate of the vertex is obtained by substituting the x-coordinate of the vertex to the quadratic expression. We get

$y = 2(2)^2 - 8(2) + 6 = -2.$

The vertex is (2, –2) and the axis of symmetry is $x = 2$.

Example 6.18

Find the intercepts of the graph of the function $y = 2x^2 - 8x + 6$

Solution

When $x = 0$, $y = 2(0)^2 - 8(0) + 6 = 6$

The y-intercept is (0, 6)

When $y = 0$, $0 = 2x^2 - 8x + 6$

We therefore solve the quadratic equation for the values of x

$2x^2 - 8x + 6 = 0.$

$2x^2 - 6x - 2x + 6 = 0.$

$2x(x - 3) - 2(x - 3) = 0.$

$(2x - 2)(x - 3) = 0.$

Either $2x - 2 = 0$ or $x - 3 = 0$.

$x = 1$ *or* $x = 3$.

The x-intercepts are (1,0) or (3,0)

The vertex

The coefficients are $a = 2$, $b = -8$ and $c = 6$

The x-coordinate of the vertex is

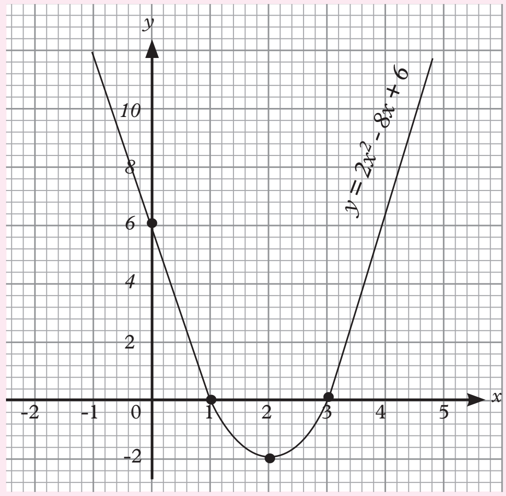
$h = -\frac{b}{2a} = -\left(\frac{-8}{2(2)}\right) = \frac{8}{4} = 2$

The y-coordinate of the vertex is obtained by substituting the x-coordinate of the vertex to the quadratic expression. We get

$y = 2(2)^2 - 8(2) + 6 = -2$

The vertex is (2, –2)

The graph is as below.



Graph 6.19

Example 6.19

Sketch the graph of $y = x^2 - 3x + 2$

Solution

The intercepts.

When $x = 0$, $y = 2$. The y-intercept is (0, 2)

When $y = 0$, then $x^2 - 3x + 2 = 0$

Solving the quadratic expression,

$x^2 - 3x + 2 = 0.$

$x^2 - 2x - x + 2 = 0.$

$x(x - 2) - 1(x - 2) = 0.$

$(x - 1)(x - 2) = 0.$

$x = 1$ *or* $x = 2$.

(1,0) and (2,0) are x-intercepts.

The vertex

$y = x^2 - 3x + 2.$

$a = 1, b = -3, c = 2.$

The x-coordinate of the vertex is

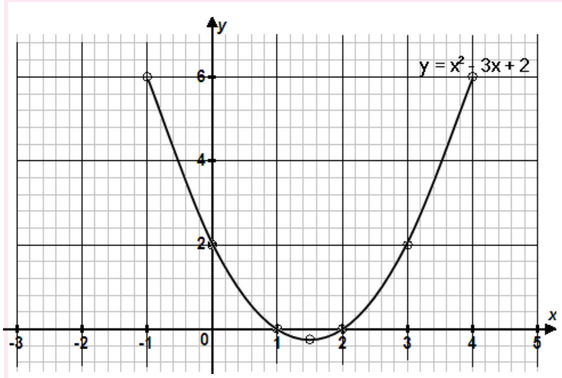
$x = -\frac{b}{2a} = \frac{-(-3)}{2(1)} = \frac{3}{2}$

The y coordinate of the vertex is

$y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 2 = \frac{9}{4} - \frac{9}{2} + 2 = -\frac{1}{4}$

The vertex is $\left(\frac{3}{2}\ \ \frac{-1}{4}\right)$.

We can now sketch the graph as long as we have the required points.



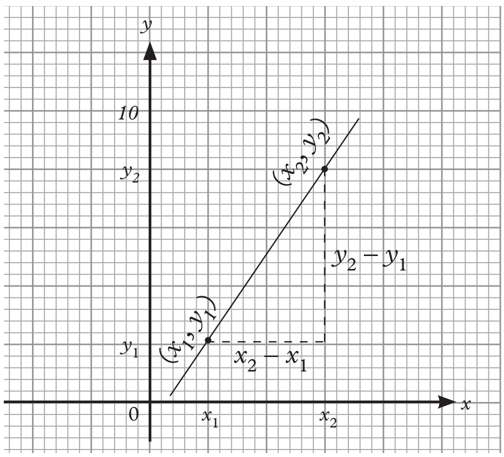
Graph 6.20

Exercise 6.8

1. Sketch the following graphs without tables of values.
   (a) $y = 2x^2 + 11x + 5$
   (b) $y = 6x^2 + 7x + 2$
   (c) $y = 5x^2 - 7x + 2$
2. Consider the quadratic function
   $y = 2x^2 + 6x + 3$
   (a) Find the intercepts with axes
   (b) Find the vertex of the function
   (c) Find the axis of symmetry.
   (d) Sketch the graph of the function.
3. Without tables of values, state the vertices, intercepts with axes, axes of symmetry, and sketch the graphs.
   (a) $y = 2x^2 + 5x - 1$
   (b) $y = 3x^2 + 8x - 6$
   (c) $2x^2 - 7x - 15 = 0$
   (d) $y = 3 + 4x - 2x^2$

Unit Summary

- **Linear function** is of the form $y = mx + c$ where m = gradient and c = y-intercept.
  Examples of linear functions are: $y = 2x - 1, y = 8, y = 7 - 7x$
- Gradient of a straight line: For line joining two points as shown in the figure.



Graph 6.21

Gradient of the line is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

- Parallel condition: When lines are parallel, they have the same gradient i.e. Consider two lines $y = m_1x + c_1$ and $y = m_2x + c_2$. These lines are parallel only and only if $m_1 = m_2$

- Perpendicular condition: When lines are perpendicular, the product of their gradients is –1 i.e.

  Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are said to be perpendicular if the product of their gradients gives –1. i.e. $m_1 \times m_2 = -1$.
- Equation of a straight line: The equation of a line $y = mx + c$ can be obtained when it passes through one point and gradient is given or when it passes through two given points.
- Quadratic function: The expression $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$, is called a quadratic function of x or a function of the second degree (highest power of x is two).
- Axis of symmetry: A quadratic function has axis of symmetry $x = h$. The axis of symmetry is parallel to the y - axis.
- Vertex of a quadratic function: Every quadratic function has vertex. The graph turns at its vertex. The vertex is the coordinate $([h, f(h))$ where $x = h$ is the axis of symmetry.

  For the expression $y = ax^2 + bx + c$, if the coefficient of the x^2 term is positive, the vertex will be the lowest point on the graph, the point at the bottom of the "∪"-shape. If the coefficient of the term x^2 is negative, the vertex will be the highest point on the graph, the point at the top of the "∩"-shape.

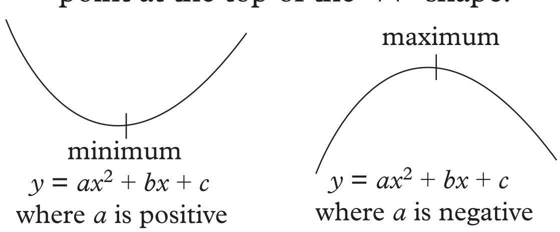


Fig. 6.3

- Intercepts of a quadratic function: The intercepts with axes are the points where a quadratic function cuts the axes.

  There are two intercepts i.e. x-intercept and y-intercept. x-Intercept for any quadratic expression is calculated by letting $y = 0$ and y-intercept is calculated by letting $x = 0$

Unit 6 Test

1. Show that the equation of the straight line passing through $(0, p)$ and $(p, 0)$ is $y + x = p$.
2. Given the function $f(x) = -2x^2 + 4x - 6$.
   (a) Identify the function and explain why.
   (b) Find the vertex of the function.
   (c) Find the intercepts of the function with axes.
   (d) Sketch the graph of the function on a Cartesian plane.
3. Consider the function

   $f(x) = 2(x - 3)(x + 1)$.
   (a) Is the curve open up or open down? Explain.
   (b) Find the vertex and intercepts of the curve.
   (c) What is the axis of symmetry of the curve?
   (d) Sketch the curve on a Cartesian plane.
4. Sketch the graph of $y = -(x + 4)(x - 9)$
5. Find the equation of the line that is parallel to another line whose equation is $4y + 5x = 6$ and passes through the point $(8, 5)$.

6. (a) Show that the point (–1, –4) lies on the line $y = 3x - 1$.

   (b) Find the equation of the line that is parallel to another line whose equation is $x + 2y + 8 = 0$ and passes through the point (–2, –3).

7. (a) Given that the line $y = 3x + a$ passes through (1, 4), find the value of a.

   (b) Sketch the graph of $y = -2x^2 - 6x - 9$.

7 COMPOUND INTEREST, REVERSE PERCENTAGE AND COMPOUND PROPORTIONAL CHANGE

Key Unit Competence: By the end of this unit, learners should be able to solve problems involving compound interest, reverse percentage and proportional change using multipliers.

Unit Outline

- Reverse percentage.
- Compound interest.
- Compound proportional change.

Introduction

Unit Focus Activity

Lucie, a farmer from the village has saved 1 000 000 FRW from her tea and livestock farming. She wants to invest the money in financial institution for three years to safeguard it and also get some interest. She visited two different financial institutions to get advice. The first institution told her that she could invest her money with them at 10% p.a simple interest for the 3 years. The second institution told her that she could invest her money with them at 10% p.a compound interest for the 3 years.

Unfortunately, Lucie did not fully understand the difference and so she does not know the best option to take. She has come to you for proper advice in regard to:

(a) What is (i) simple interest (ii) compound interest?

(b) Which of the two types of investment better?

(c) What would be the difference in the total interest generated through the two types of investments after the 3 years?

By performing the necessary calculations, kindly write down on a piece of paper your full advice to Lucie regarding her three queries.

7.1 Reverse Percentage

Activity 7.1

Consider an iron box that costs 450 FRW after a 25% increase in its original price.

1. Explain how you can determine the original price of the iron box. Then determine that price.
2. Compare the current price and the original price of the iron box. Which is higher? Why?

Reverse percentage involves working out the original quantity of an item backwards after the increase or decrease in its quantity. This method is applied when given a quantity after a percentage increase or decrease and one is required to find the original quantity.

Example 7.1

A radio is sold at 620 FRW after a 40% increase in the price. Find the original price.

Solution

What is required here is to reverse the process of raising the price by the given percentage.

Let the original; price be x

$$\text{The new price} = (100 + 40)\% \text{ of } x$$
$$= 140\% \text{ of } x$$
$$140\% \text{ of } x = 620 \text{ FRW}$$
$$\frac{140}{100}x = 620 \text{ FRW}$$
$$x = \frac{620 \times 100}{140}$$
$$= 442.86 \text{ FRW}$$

Example 7.2

For purposes of sales promotion, the price of a book has been reduced by 20% to 3 600 FRW. What was the price before the reduction?

Solution

We are required to use the reverse process of decreasing the price by the given percentage.

Let the old price by y

$$\text{Then the new price} = (100 - 20)\% \text{ of } y$$
$$= 80\% \text{ of } y$$
$$80\% \text{ of } y = 3\,600 \text{ FRW}$$
$$\frac{80}{100}y = 3\,600 \text{ FRW}$$
$$y = \frac{3\,600 \times 100}{80}$$
$$= 4\,500 \text{ FRW}$$

Alternative Method

Old price = FRW y which is 100%

New price = 3 600 FRW which is 80%

$y : 3\,600 = 100 : 80$

$$\frac{y}{3\,600} = \frac{100}{80} \quad \text{(a proportion)}$$
$$y = \frac{100}{80} \times 3\,600 \text{ FRW}$$
$$y = 4\,500 \text{ FRW}$$

Old price = 4 500 FRW

Exercise 7.1

1. A man's daily wage was increased by 25% to 500 FRW. Find how much it was before the increase.
2. The price of an article is decreased by 5% to 1 900 FRW. What was the price before the decrease?
3. A company produced 23 000 shirts in September. This was 8% less than the August production. How many shirts did the company produce in August?
4. A new car falls in value by 30% a year. After a year, it is worth 84 000 FRW. Find the price of the car when it was new.

7.2 Compound interest

7.2.1 Definition of compound interest

Activity 7.2

Find out from reference books or the internet the:

1. definition of interest and some of the area of its application.
2. ways of calculating compound interest.
3. difference between compound interest and simple interest.

When money is borrowed from or deposited in a financial institution, it earns an interest at the end of each interest period as specified in the terms of investment, for example at the end of each year, half year etc. Instead of paying the interest to the owner, it is added to the principal at the end of each period i.e. compounded with the principal. The resulting amount is then taken to be the principal for the next interest period. The interest earned in this period is higher than in the previous one. The interest so earned is called compound interest.

Therefore, compound interest defines the interest calculated on the initial principal and also on the accumulated interest of previous periods of a deposit or loan. There are two ways of calculating compound interest:

(a) the step by step method

(b) the compound interest formula.

7.2.2 Step by Step Method

Activity 7.3

Using a step by step method, determine to amount accumulated by a principal of 500 000 FRW invested at 8% compound interest for 4 years. What is the total interest realised?

Compound interest can be calculated step by step through compounding the interest generated with the principal.

The interest is

$$I = \text{Principal} \times \frac{\text{Rate}}{100} \times \text{time} = \frac{P \times R \times t}{100}$$

Example 7.3

Find the total amount of money accumulated after 3 years if 10 000 FRW is invested at 10% p.a. compound interest.

Solution

Since the principal changes at the beginning of every year, we calculate the interest at the end of each year.

Interest for year 1 $= \frac{10\,000 \times 10 \times 1}{100}$

$= 1\,000$ FRW

Amount after year 1

$= P + I$

$= (10\,000 + 1\,000)$ FRW

$= 11\,000$ FRW

Now 11 000 FRW becomes the principal for 2nd year.

Interest for year 2 $= \frac{11\,000 \times 10 \times 1}{100}$

$= 1\,100$ FRW.

Amount after year 2

$= P + I$

$= 11\,000$ FRW $+ 1\,100$ FRW

$= 12\,100$ FRW

12 100 FRW becomes our principal for 3rd year

Interest for year 3

$= \frac{12\,100 \times 10 \times 1}{100}$ FRW

$= 1\,210$ FRW

Amount after year 3

$= P + I$

$= (12\,100 + 1\,210)$ FRW

$= 13\,310$ FRW

Example 7.4

Find the compound interest earned on 90 000 FRW for 3 years at 7% p.a.

Solution

Interest for year 1 $= \frac{90\,000 \times 7 \times 1}{100}$ FRW

$= 6\,300$ FRW

Amount after year 1 $= 90\,000 + 6\,300$

$= 96\,300$ FRW

Interest for year 2 $= \left(\frac{96\,300 \times 7 \times 1}{100}\right)$ FRW

$= 6\,741$ FRW

Amount after year 2 $= 96\,300 + 6\,741$

$= 103\,041$ FRW

Interest for year 3 $= \frac{103\,041 \times 7 \times 1}{100}$

$= 7\,212.87$ FRW

Amount after year 3 = 103 041 + 7 212.87

= 110 253.87 FRW

Compound interest

= (110 253.87 – 90 000) FRW

= 20 253.87 FRW

Example 7.5

Daka borrows 3 800 FRW from Jane at 10% p.a. compound interest. At the end of each year, he pays back 910 FRW. How much does he owe Jane at the beginning of the third year?

Solution

Interest for year 1 = $\frac{3\ 800 \times 10 \times 1}{100}$

= 380 FRW

Amount after year 1 = (3 800 + 380) FRW

= 4 180 FRW

Less payment made after year 1: – 910 FRW

= 3 270 FRW

Interest for year 2 = $\frac{3\ 270 \times 10 \times 1}{100}$

= 327 FRW

Amount after year 2 = 3 270 + 327 FRW

= 3 597 FRW

Less payment made after year 2: – 910 FRW

= 2 687 FRW

Therefore Jane is owed 2 687 FRW at the beginning of the third year.

Example 7.6

Find the accumulated amount of money after $1\frac{1}{2}$ years, for 10 500 FRW, invested at the rate of 8% p.a. compounded semi-annually.

Solution

P = 10 500 FRW, R = 8% p.a.,

T = $1\frac{1}{2}$ years

There are 3 half years in one and a half years

The rate of interest for each half year is 4% (i.e. 8% ÷ 2) while the interest period is each $\frac{1}{2}$ year.

I^{st} half year interest = $\frac{10\ 500 \times 4 \times 1}{100}$ FRW

= 420 FRW

Amount after 1^{st} half = 10 500 + 420

= 10 920 FRW

Interest after 2nd half-year = $\frac{10\ 920 \times 4 \times 1}{100}$

= 436.80 FRW

Amount after 1 year

= (10 920 + 436.80) FRW

= 11 356.80 FRW

Interest after 3rd half - year

= $\frac{11\ 356.80 \times 4 \times 1}{100}$ FRW

= 454.27 FRW

Amount after $1\frac{1}{2}$ years

= 11 356.80 + 454.27

= 11 811.07 FRW

= 11 811.07 FRW

Exercise 7.2

1. Find the amount and the compound interest for each of the following correct to the nearest FRW:
   - **(a)** 8 000 FRW invested for 3 years at 4% p.a. compound interest.
   - **(b)** 48 000 FRW invested for 2 years at 6% p.a. compound interest.
   - **(c)** 36 000 FRW invested for 2 years at 5% p.a. compound interest.

2. Determine the difference between the simple interest and compound interest earned on 15 000 FRW for 2 years at 3% p.a.
3. Kampire invested P FRW, which amounted to 8 420 FRW in 3 years at a rate of 4.5% p.a. compound interest. Find the value of P.
4. Determine the compound interest earned on 45 000 FRW after 3 years at the rate of 6% p.a.
5. Mugisha borrows a sum of 8 000 FRW at 10% p.a. simple interest and lends that to Neza at the same rate compound interest. How much will Mugisha gain from this transaction after 3 years?

7.2.3 The compound interest formula

Activity 7.4

1. Consider the case in which 10 000 FRW is invested in a bank for 3 years at the rate of 5% p.a. compound interest.
2. Using the step by step method calculate the amount of money accumulated after every year.

Assuming 5 000 FRW is the principal amount compounded at 6% p.a.The amount of money accumulated after every year is calculated as follows:

Amount after 1st year

$= 5\,000 \text{ FRW} + \frac{5\,000 \times 6 \times 1}{100}$

$= 5\,000 \text{ FRW} + 5\,000 \times \frac{6}{100}$

$= 5\,000 \text{ FRW } (1 + \frac{6}{100})$

Amount after 2nd year

$= 5\,000\left(1+\frac{6}{100}\right)+5\,000\left(1+\frac{6}{100}\right)\times\frac{6}{100}\times 1$

$= 5\,000\left(1 + \frac{6}{100}\right)^2$

Amount after 3rd year

$= 5\,000\left(1+ \frac{6}{100}\right)^2+5\,000\left(1\frac{6}{100}\right)^2\times\frac{6}{100}\times 1$

$= 5\,000\left(1 + \frac{6}{100}\right)^3$

Amount after 4th year

$= 5\,000\left(1\frac{6}{100}\right)^3 + 5\,000\left(1 + \frac{6}{100}\right)^3 \times \frac{6}{100}$

$= 5\,000\left(1 + \frac{6}{100}\right)^4$

Amount after n years

$= 5\,000\ (1 + \frac{6}{100})^n$ FRW

Considering the case in which P is invested in a bank for n-interest periods at the rate of r% p.a. The accumulated amount (A) after the given time is given by:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

where n is the number of interest periods.

This is called the compound interest formula. It is conveniently used in solving problems of compound interest especially those involving long periods of investments or payment.

In this method, the accrued compound interest is obtained by subtracting the original principal from the final amount.

Thus, Compound interest = Accumulated amount (A) – Principal (P)

Note that the principal and the interest earned increased after each interest period. We can also deduce that;

Compound interest = Accumulated amount - Principal amount

$$I = A - P$$

Accumulated amount	=	Principal amount	+	Compound interest

$$A = P + I$$

Example 7.7

A trader deposited 63 000 FRW in a fixed deposit account with a local bank which attracted an interest of 8% p.a. compound interest. Find:

(a) the total amount after 4 years;

(b) compound interest.

Solution

$P = 63\ 000$ FRW $n = 4, r = 8\%$ p.a.

$A = P\left(1 + \frac{r}{100}\right)^n$

(a) $A = 63\ 000\left(1 + \frac{8}{100}\right)^4$

$= 63\ 000\ (1.08)^4$

$= 63\ 000 \times 1.36048896$

$= 85\ 710.80$ FRW

(b) $I = (85\ 710.80 - 63\ 000)$ FRW

$= 22\ 710.80$ FRW

$= 63\ 000 \times 1.36048896$

$= 85\ 710.80$ FRW

Example 7.8

Find the accumulated amount that 30 000 FRW will yield if deposited for 2 years at 6% p.a. compounded semi-annually.

Solution

For every year there will be two interest periods;

Thus, in 2 years we will have (2 × 2) interest periods i.e. n = 4

The rate per interest period = 6 ÷ 2 = 3%

Thus, $A = 30\ 000$ FRW $\left(1 + \frac{3}{100}\right)^4$

$= 30\ 000$ FRW $(1.03)^4$

$= 30\ 000$ FRW $\times 1.12550881$

$= 33\ 765.26$ FRW

$= 33\ 765$ FRW

Example 7.9

Find the compound interest earned on FRW 15 000 invested for 3 years, at 20% p.a. compounded quarterly.

Solution

Here, each year has 4 interest periods (quarterly) i.e. in 3 years, there are 12 interest periods (3 × 4 = 12).

The rate, r % = 20 ÷ 4 = 5% p.a.

$A = 15\ 000\left(1 + \frac{5}{100}\right)^{12}$

$= 15\ 000$ FRW $(1.05)^{12}$

$= 26\ 937.84$ FRW

$I = 26\ 937.84 - 15\ 000$

$= 11\ 937.84$ FRW

$= 11\ 938$ FRW

Example 7.10

A trader bought a posho mill worth 155 000 FRW from a dealer. He paid a deposit of 10% of the total cost of the posho mill and the remaining amount plus accrued compound interest was repaid in equal monthly instalments for 3 years at a rate of 6% p.a. Calculate the amount of money the trader paid per month to the nearest shilling.

Solution

The amount of deposit $= \frac{10}{100} \times 155\ 000$

$= 15\ 500$ FRW

Amount charged = cost price less deposit

$= 155\ 000 - 15\ 500$

$= 139\ 500$ FRW

Amount $= 139\,500\,(1 + \frac{6}{100})^3$

$= 139\,500\,(1.06)^3$ *FRW*

$= 16\,614.67$

Monthly instalment $= \frac{16\,614.67}{36}$

$= 4\,615.20$ *FRW*

$= 4\,615$ *FRW*

Exercise 7.3

1. Determine the accumulated amount for each of the following:
   (a) 19 000 FRW invested for 2 years at 3% p.a. compound interest.
   (b) 8 300 FRW invested for 3 years at 4% p.a. compound interest.
2. A businesswoman invested 4 500 FRW for 2 years in a savings account. She was paid 4% per annual compound interest. How much did she have in her savings account after 2 years?
3. Adams invests 4 500 FRW at a compound interest rate of 5% per annum. At the end of n complete years the investment has grown to 5 469.78 FRW. Find the value of n.
4. A company bought a car that had a value of 12 000 FRW. Each year the value of the car depreciates with 25%. Work out the value of the car at the end of three years.
5. Erick invested 60 000 FRW for 3 years in a savings account. He gets a 3% per annum compound interest. How much money will Erick have in the savings account at the end of 3-years?

7.3 Compound proportional change

Activity 7.5

Consider that 3 people working at the same rate can plough 2 acres of land in 3 days.

What do you think will happen if the working days are increased to five working at the same rate? Discuss.

Sometimes, a quantity may be proportional to two or more other quantities. In such a case, the quantities are said to be in **compound proportion**.

Problems involving rates of work, and other similar problems, often contain quantities that are in compound proportion. Such problems are solved using either the ratio or unitary method.

Example 7.11

Eighteen labourers dig a ditch 80 metres long in 5 days. How long will it take 24 labourers to dig 64 metres long? What assumptions have you made?

Solution

Notice that the number of days depends on the number of labourers as well as on the length of the ditch. Thus, this is a problem in compound proportion.

1. Ratio method

Length of ditch decrease from 80 m to 64 m, i.e. in the ratio 64 : 80.

A shorter ditch takes a shorter time.

$\therefore$ *multiply days by* $\frac{64}{80}$.

Number of labourers increases from 18 to 24, i.e. in the ratio 24 : 18.

More labourers take a shorter time.

$\therefore$ *multiply days by* $\frac{18}{24}$.

80 metres of ditch are dug by 18 labourers in 5 days.

64 metres of ditch are dug by 24 labourers in

$5 \times \frac{64}{80} \times \frac{18}{24}$ *days* $= 3$ *days.*

2. Unitary method

18 labourers dig 80 metres of ditch in 5 days.

∴ *1 labourer digs 80 metres of ditch in (5 × 18) days = 90 days.*

∴ *1 labourer digs 1 metre of ditch in (90 ÷ 80) days = $1\frac{1}{8}$ days.*

∴ *1 labourer digs 64 metres of ditch in ($1\frac{1}{8}$ × 64) days = 72 days.*

∴ *24 labourers dig 64 metres of ditch in (72 ÷ 24) days = 3 days.*

The assumptions made in this calculation are:

1. *The width and depth of the ditch is uniform.*
2. *The hardness of the ground is the same throughout.*
3. *The labourers work at the same rate.*

Example 7.12

To plant a certain number of tree seedlings Mutoni takes 5 hours. Gahigi takes 7 hours to plant the same number of seedlings. If Mutoni and Gahigi worked together, how long would they take to plant the same number of seedlings?

Solution

In 1 hour, Mutoni plants $\frac{1}{5}$ of the seedlings.

In 1 hour, Gahigi plants $\frac{1}{7}$ of the seedlings.

∴ *in 1 hour, Mutoni and Gahigi together plant $\frac{1}{5} + \frac{1}{7} = \frac{12}{35}$ of the seedlings.*

$\frac{12}{35}$ of the seedlings are planted in 1 hour.

∴ *$\frac{12}{35}$ of the seedlings are planted in*

$(\frac{35}{35} \div \frac{12}{35})$ *hours* $= \frac{35}{12}$*hours.*

∴ *Mutoni and Gahigi working together take $\frac{35}{12}$ hours*

= 2 hours 55 minutes.

Note: We first found out the fraction of the number of seedlings that each person plants in 1 hour, then the fraction of the number of seedlings that they plant together in 1 hour.

Invert this fraction, and the result is the number of hours they take working together.

Trees are life.

If we clear our forests, we will have drought continously.

Let us plant more trees to conserve our environment and to preserve our lives.

Exercise 7.4

1. 16 men dig a trench 92 m long in 9 days. What length of trench can 12 men dig in 15 days?
2. To unclog a silted drain 85 m long, 15 workers take 10 days. Find how many workers are required to unclog a similarly silted drain 51 m long drain in 5 days.
3. Six people pay 40 320 FRW for a 7 day stay at a hotel. How much would eight people pay for a 3 day stay?
4. A car hire company with 24 cars uses 2 940 litres of petrol in 5 days. How long would 4 116 litres of petrol last if the company had 28 cars and the consumption rate does not change?
5. A transport company charges 54 800 FRW to move a load of 2.8 tonnes for 350 km. For what load will the corporation charge 47 040 FRW for 400 km?
6. A man, standing next to a railway line, finds that it takes 6 seconds for

a train, 105 m long, travelling at 63 km/h, to pass him. If another train, 100 m long, takes 5 seconds to pass him, at what speed is it moving?

7. It takes 15 days for 24 lorries, each of which carries 8 tonnes, to move 1 384 tonnes of gravel to a construction site. How long will it take 18 lorries, each of which carries 10 tonnes, to move 1 903 tonnes of the gravel?
8. A car moving at 65 km/h takes 2 h 24 min to travel 156 km. What distance does the car travel in 48 min moving at 55 km/h?
9. Twelve men, working 8 hours a day, can do a piece of work in 15 days. How many hours a day must 20 men work in order to do it in 8 days?

Unit Summary

- Reverse percentage is the working out of original price of a product backwards after the increase.
- Compound interest: is the interesst calculated on both the amount borrowed and any accumulated previous interests.
- The compound interest formula states as shown:

  $A = P\left(1 + \frac{r}{100}\right)^n$ where n is the number of interest periods, r is the rate, A is the initial amount and P is the principal

Unit 7 Test

1. After the prices of fuel increased by 15%, a family's annual heating bill was 1 654 FRW. What would the bill have been without the increase in price?
2. An insurance company offers a no claim discount of 55% for drivers who have not had an accident for 4 years. If the discounted premium for such a driver is 3 340 FRW, how much did the driver save?
3. After a long-haul flight, the total weight of a passenger jet had decreased by 27% to 305 000 kg. What weight of fuel was the aircraft carrying at take off?
4. Lange borrows 16 000 FRW to buy a coloured TV set at 10% p.a. compound interest. He repays 980 FRW at the end of each year. How much does he still owe at the end of 3 years?
5. If 12 000 FRW is invested at 12% p.a. compounded quarterly, find the accumulated amount after one year to the nearest Francs.
6. Mwiza borrowed 2 000 FRW at 5% p.a. compound interest from a microfinance company. She paid back 350 FRW at the end of each year. How much does Mwila still owe the company at the end of the second year?
7. A sum of money is invested at compound interest and it amounts to 420 FRW at the end of the first year and 441 FRW at the end of the second year. Determine;
   (a) the rate in percent
   (b) the sum of money invested.
8. Calculate the amount after 3 years if 7 800 FRW is invested at $12\frac{1}{2}\%$ p.a. compound interest (give your answer to the nearest francs).
9. If you deposit 4 000 FRW into an account paying 6% annual interest

compounded quarterly, how much money will be in the account after 5 years?

10. If you deposit 6 500 FRW compounded monthly into an account paying 8% annual interest compounded monthly, how much money will be in the account after 7 years?

11. If you deposit 8 000 FRW into an account paying 7% annual interest compounded quarterly, how long will it take to have 12 400 FRW in the account?

12. One pipe can fill a bath in 5 minutes and another can empty the same bath in 10 minutes. Both pipes are opened at the same time and after 5 minutes, the second pipe is turned off. What fraction of the bath is then full? How long will it take for the first pipe to fill the bath completely from then?

13. A man can do a job in $4\frac{1}{2}$ days. Another man can do the same job in 9 days. How long will the two men take on the job if they work together?

14. Workmen A and B, working together, do a certain job in 1 hour. Workman A alone does the job in 3 hours. How long does it take workman B alone to do the job?

15. Working alone, Alex can do some job in 6 days. John, also working alone, can do the same job in 9 days. Alex starts alone, but is joined by John after 1 day. How long do they take to finish the work together?

16. Three persons, Peter, Mike and James, build a certain length of wall in 2 days. Peter and Mike together could build the same length of wall in 4 days, and Mike and James together would take 3 days.

(a) Find the fraction of the length of the wall that Mike and James build in 2 days and hence find how long Peter would take by himself.

(b) Find the fraction of the length of the wall that Peter and Mike build in 2 days and hence find how long James would take by himself.

(c) Similarly, find how long Mike would take by himself.

8 RIGHT-ANGLED TRIANGLES

Key unit competence

By the end of the unit, learners should be able to find the length, sides and angles in right-angled triangles using trigonometric ratios.

Unit Outline

- Definition of a right-angled triangle.
- Elements of a right-angled triangle.
- Relationship between the elements of a right-angled triangle through the use of Pythagoras theorem.
- Median and perpendicular heights.
- Determining the sides of right-angled triangle given their orthogonal projections on the hypotenuse.
- Solving problems in right-angled triangles using the element properties and the Pythagoras theorem
- Trigonometric ratios in a right-angled triangle: sine, cosine and tangent.

Introduction

Unit Focus Activity

1. Fig. 8.1 below shows a common roofing truss.

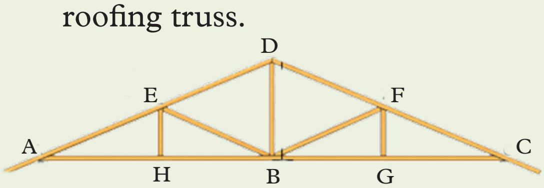


Fig 8.1: Common roofing truss

Given that ED = $\frac{1}{2}$ AD = 5cm, ∠BAD = 30^0, find length:

(a) EB (b) AB (c) EH

In case, state the theorem or the mathematics relation you have applied.

2. Fig. 8.2 shows another common design of a roofing truss.

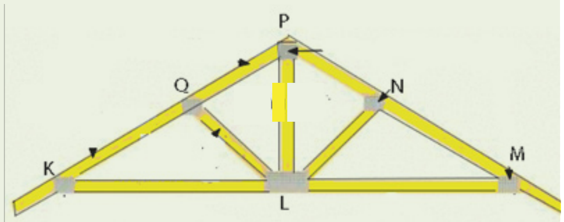


Fig. 8.2

Section KLP, forms a right-angled triangle. Line LQ is the altitude of triangle KLP i.e ∠LQP = 90°. Given that LQ = 6 cm and KQ = 8 cm, find the length of :

(a) QP (b) KL

In each case, state the theorem or the Mathematics relation you have applied.

8.1 Review of Pythagoras theorem

Activity 8.1

1. Draw and label all features and sides of a right angled triangle.
2. Show the relationship between the three sides of a right angled triangle.

In a right-angled triangle, one angle is 90°. The longer side in a right-angled triangle called the **hypotenuse** and the two shorter sides (**legs**).

Consider the right-angled triangle in Fig 8.3 below.

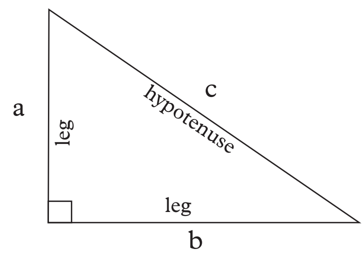


Fig. 8.3

Pythagoras theorem states that the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.

Pythagoras theorem

$a^2 + b^2 = c^2$

Example 8.1

In Fig 8.4 below, work out the missing measurements on the right angled triangles

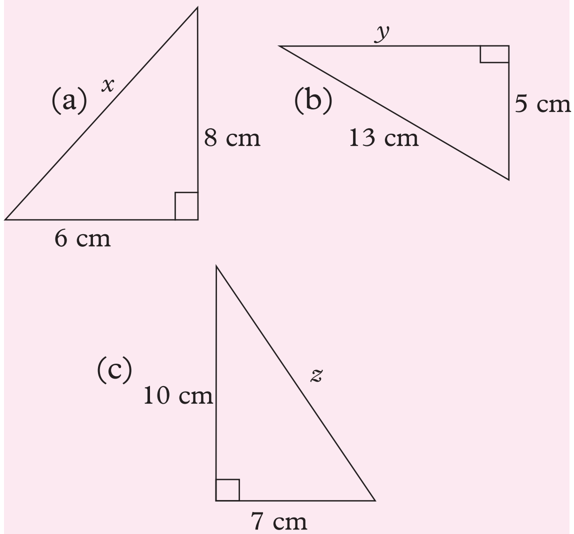


Fig. 8.4

Solution

(a) Triangle (a)

Use Pythagoras theorem

$a^2 + b^2 = c^2$

Let $a = 6$ cm, $b = 8$ cm; $c = x$ cm

$6^2 + 8^2 = x^2$

$36 + 64 = x^2$

$100 = x^2$

Finding the square root of 100

$x = \sqrt{100}$ cm

$x = 10$ cm

(b) Triangle (b)

Use Pythagoras theorem

$a^2 + b^2 = c^2$

$a = 5$ cm, $b = y$ cm and $c = 13$ cm

$b^2 + y^2 = 13^2$

$25 + y^2 = 169$

$y^2 = 169 - 25$

$y^2 = 144$

Finding the square root of 144

$y = \sqrt{144}$ cm

$= 12$ cm

(c) Triangle (c)

Use Pythagoras theorem

$a^2 + b^2 = c^2$

$a = 7$ cm, $b = 10$ cm and $c = z$ cm

$7^2 + 10^2 = z^2$

$49 + 100 = z^2$

$z^2 = 149$

Finding the square root of 149

$z = \sqrt{149}$ cm

$= 12.21$ cm (2dp)

Exercise 8.1

1. Determine, to two decimal places, the length of the third side of the right angled triangle where the one side and the hypotenuse have given below.
   (a) 5 cm, 12 cm
   (b) 1 cm, 2 cm
   (c) 3 cm, 4 cm
   (d) 2.5cm, 3 cm
   (e) 1 cm, 1.73 cm
   (f) 2 cm, 7 cm
2. The diagram in Fig 8.5 shows a wooden frame that is to be part of the roof of a house.

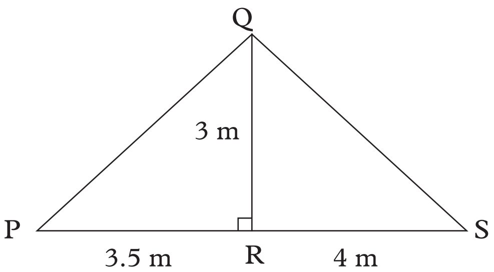


Fig. 8.5

(a) Use Pythagoras theorem in triangle PQR to find the length of PQ

(b) Calculate the length QS

3. Fig 8.6 below shows an isosceles triangle with a base of length 4 m and perpendicular height 8 m. Calculate length labeled x of one of the equal sides.

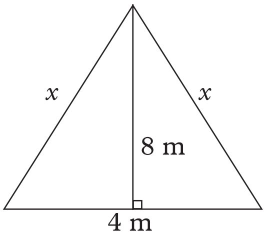


Fig. 8.6

4. The diagram in Fig 8.7 shows a vertical flagpole of height 5.2 m with a rope tied to the top. When the rope is pulled tight, the bottom end is 3.8 m from the base of the flagpole. Calculate the length of the rope.

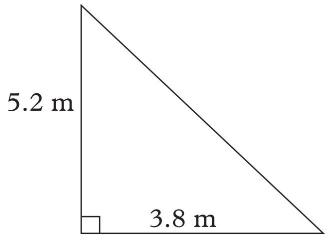


Fig. 8.7

8.2 Median theorem of a right-angled triangle

Activity 8.2

1. Using a ruler and pair of compass or protractor, draw any rectangle ABCD with measurements of your choice.
2. Draw the diagonals AC and BD and label the point of their intersection as E.
3. Measure and compare the diagonal segments BE and ED, and AE with EC. What do you notice? What can you say about the diagonals of a rectangle?
4. Consider the right angled-triangle ABD in Fig. 8.8 below isolated from rectangle ABCD.

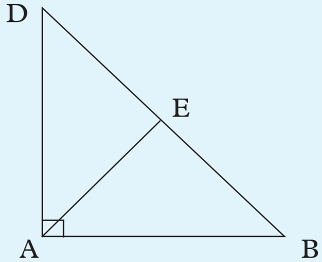


Fig. 8.8

(a) Measure and compare the segment BE and DE of the hypotenuse. What do you notice?

(b) In what proportions does point E divide the hypotenuse DB? What is the name of point E in relation to side DB?

(c) What is the name of line AE that is drawn to from vertex A to point E?

(d) Measure and compare the length of AE to the hypotenuse DB. What do you notice?

Activity 8.3

1. Draw any right-angled triangle PQR, with dimensions of your choice and $\angle Q = 90°$.
2. Measure and locate the midpoint of the hypotenuse PR and label it S. Join vertex Q to points with a straight line.
3. Measure and compare the lengths QS and the hypotenuse PR. What do you notice?
4. Measure and compare the lengths QS with PS and RS. What do you notice?.

Consider a right-angled triangle. A line drawn from any vertex of the triangle to the midpoint of the side opposite to that vertex is called a median, Fig. 8.9 below.

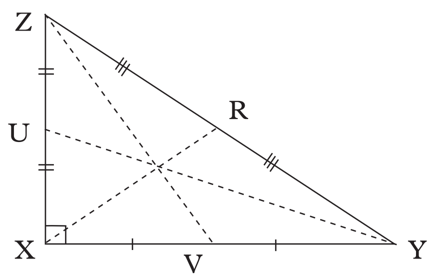


Fig. 8.9

XR, ZV and YU are the medians of triangle XYZ (Fig. 8.9).

Let us consider the median XW from the right-angled vertex X to the midpoint W of the hypotenuse YZ as shown in Fig. 8.10.

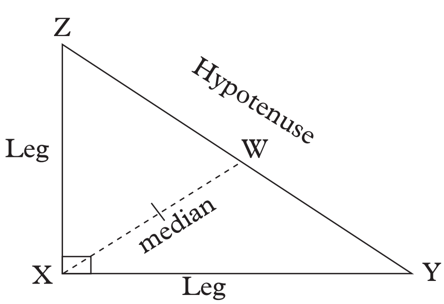


Fig. 8.10

We observe the following:

$WX = \frac{1}{2}YZ$

Hence $WX = WZ = WY$.

Theorem

The median theorem of a right-angled triangle states that: **the median from the right angled vertex to the hypotenuse is half the length of the hypotenuse.**

$$\text{Median} = \frac{1}{2}\text{ Hypotenuse}$$

As such, that median subdivides the right-angled triangle into two similar isosceles triangles.

Example 8.2

In a right-angled triangle ABC, line BC is the hypotenuse and AN is 5 cm long. What is the length of the hypotenuse?

Solution

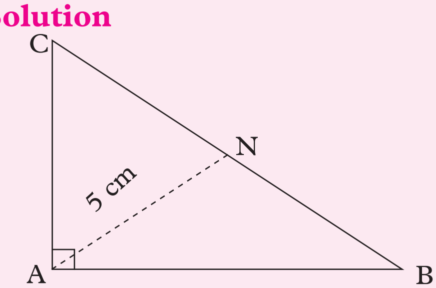


Fig. 8.11

$\text{Median } AN = \frac{1}{2}BC \text{ (hypotenuse)}$

$5 \text{ cm} = \frac{1}{2}BC$

$BC = 5 \text{ cm} \times 2 = 10 \text{ cm}$

Example 8.3

In a right-angled triangle, the median to the hypotenuse has a length of (3x – 7) cm. the hypotenuse is (5x – 4) long. Find the value x, hence find the length of the hypotenuse.

Solution

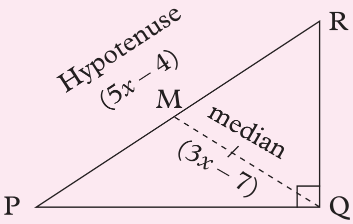


Fig. 8.12

$QM = \frac{1}{2}PR$

$3x - 7 = \frac{5x - 4}{2}$

By cross multiplication

$2(3x - 7) = 5x - 4$

$6x - 14 = 5x - 4$

$x = 10$ cm

Hence, the length of the hypotenuse is

$(5x - 4)$ cm $= (5 \times 10 - 4)$ cm

$= 46$ cm

Example 8.4

Fig. 8.13 below shows right-angled triangle PQR. QT is the median to the hypotenuse and ∠QRP = 37°. Find ∠PTQ.

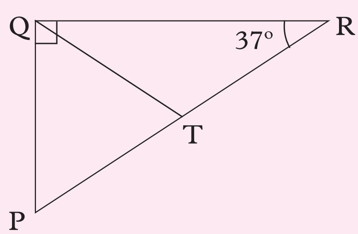


Fig. 8.13

Solution

$QT = \frac{1}{2}PR$

$QT = PT = TR$

Hence ΔPTQ and ΔQTR are isosceles

$\angle TQR = 37°$

$\therefore \angle RTQ = 180° - (37° + 37°)$

$= 106°$

$\angle PTQ = 180 - 106 = 74°$

Exercise 8.2

1. In a right-angled triangle, the median to the hypotenuse is 4.5 cm. What is the length of the hypotenuse?
2. One leg of a right triangle is 12 cm. The median to the hypotenuse is 7.5 cm. Find the:
   (a) length of the hypotenuse.
   (b) length of the other leg of the hypotenuse.
3. The two legs of a right-angled triangle are 4.5 cm and 6 cm long. Find the length of the median from the right-angled vertex to the hypotenuse.
4. Triangle KLM is right-angled at vertex L and ∠LKM = 24°. N is the midpoint of the hypotenuse KM. Find the value of angle:
   (a) KLN (b) LNM
5. In a right-angled triangle EFG, the hypotenuse is $(3x + 8)$cm long. The median to the hypotenuse is $(5x - 10)$ long. Find the value of x hence find the length of the median.

8.3 Altitude (Height) theorems of a right-angled triangle

Activity 8.4

Materials: Manila paper, protractor, ruler, pencil, a pair of compasses.

1. Draw a right-angled triangle ABC with dimensions of your choice on a manila paper.
2. Drop a perpendicular from the right-angled vertex C to intersect the hypotenuse at point N. Line NC is known as the altitude of the triangle.
3. Cut off triangles ANC and NBC from the manila paper.
4. Rotate the triangular cut out NBC by 90° anticlockwise.
6. Test triangles ANC and CNB for similarity using equality of corresponding angles. What do you notice?

Activity 8.5

Use the two triangular cut outs you prepared in activity 8.4, Fig. 8.14 below.

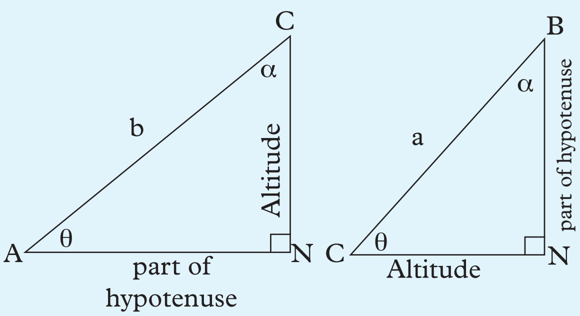


Fig. 8.14

1. Measure the length of the altitude CN and each of the two segments of the hypotenuse i.e. AN and NB.
2. Determine and compare the ratios of the corresponding sides $\frac{NC}{AN}$ and $\frac{NB}{NC}$. What do you notice?

A line drawn from any vertex of a triangle to intersect the side opposite to the vertex perpendicularly is known as an altitude of the triangle.

The altitude from the right-angled vertex of a right-angled triangle to the hypotenuse divides the triangle into two similar triangles. The corresponding those two angles are equal.

For example, triangle EFG(*Fig 8.15*) forms two similar triangle, EHG and GHF, when cut along the altitude GH as shown in Fig. 8.15.

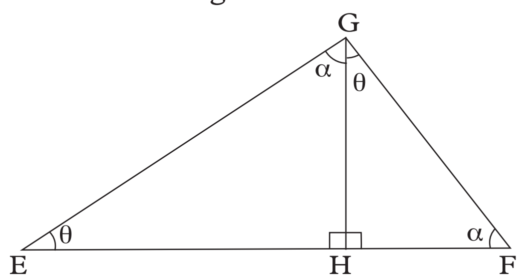


Fig. 8.15

$\angle GEH = \angle FGH = \theta°$

$\angle EHG = \angle GHF = 90°$

$\angle EGH = \angle GFH = \propto°$

Consider a right-angled triangle EFG with the altitude drawn from the right-angled vertex G to the hypotenuse.

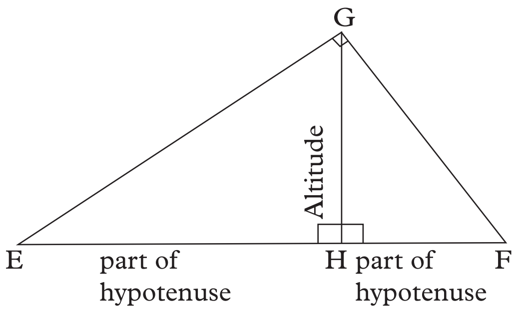


Fig. 8.16

The ratio

$$\frac{HG\ (Altitude)}{EH\ (part\ of\ hypotenuse)} = \frac{HF\ (Other\ part\ of\ hypotenuse)}{HG\ (Altitude)}$$

Or simply put

$$\frac{altitude}{part\ of\ hypotenuse} = \frac{other\ part\ of\ hypotenuse}{altitude}$$

This is the mathematical representation of the altitude theorem of a right-angled triangle.

Thus, the altitude theorem of a right-angled triangle states that,

"The altitude to the hypotenuse of a right-angled triangle is the mean proportional between the segments into which it divides the hypotenuse."

$$\frac{Altitude}{Part\ of\ hypotenuse} = \frac{Other\ part\ of\ hypotenuse}{Altitude}$$

Example 8.5

In a right-angled triangle ABC, AD is the altitude from vertex A to the hypotenuse. If AD = 6 cm and DC = 9 cm, find the length of segment BD

Solution

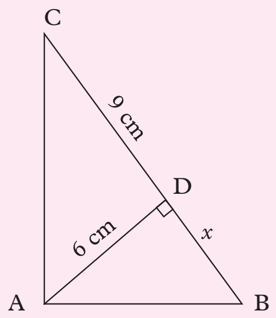


Fig. 8.17

From the altitude theorem

$$\frac{AD}{DC} = \frac{BD}{AD}$$

$$\frac{6}{9} = \frac{x}{6}$$

$$\Rightarrow 9x = 36$$

$$x = \frac{36}{9} = 4$$

$BD = 4\ cm$

Example 8.6

Fig 8.18 shows a right-angled triangle ΔPQR in which PM = 12 cm and MQ = 3 cm. find the

(a) *Length of MR.*

(b) *Area of Δ PQR.*

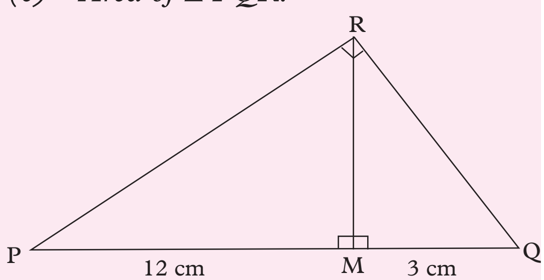


Fig 8.18

Solution

(a) *In triangle PQR, PQ is the hypotenuse an MR is the altitude to* *the hypotenuse. Applying the altitude theorem,*

$$\frac{MR}{PM} = \frac{MQ}{MR}$$

Let $MR = x$

$$\Rightarrow \frac{x}{12} = \frac{3}{x} \Rightarrow x^2 = 12 \times 3$$

$$\therefore \quad x = \sqrt{36}$$

$$= 6 \text{ cm}$$

Hence $MR = 6$ *cm*

(b) *Area of* ΔPQR

$$= \frac{1}{2} \text{ base} \times \text{height}$$

$$= \frac{1}{2} PQ \times MR$$

$$= \frac{1}{2}(12 + 3) \times 3 \text{ cm}^2 = \frac{1}{2} \times 45 \text{ cm}^2$$

$$= \frac{45}{2} \text{ cm}^2$$

Exercise 8.3

1. Find the length AD in triangle ABC in Fig. 8.19 given that DC = 10 cm and DB = 8 cm.

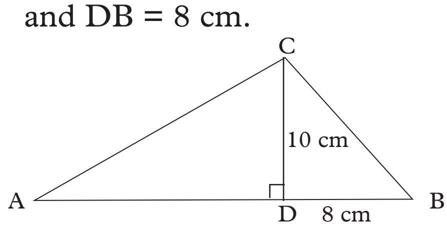


Fig 8.19

2. Find the length MP in triangle MNO shown in Fig. 8.20 given that OP = 4 cm and NP = 12 cm.

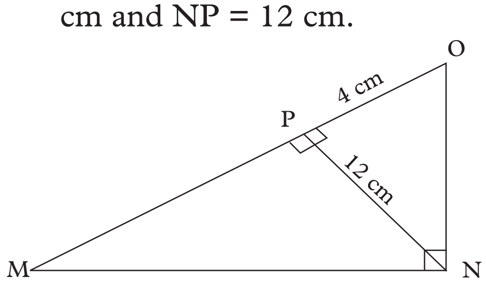


Fig 8.20

3. In the right-angled triangle EFG shown in Fig. 8.21, EH = 4 cm and HF = 9 cm. Find the altitude to the hypotenuse.

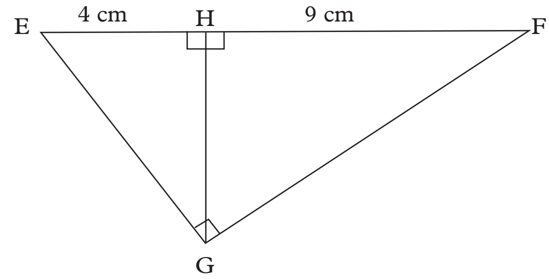


Fig 8.21

4. In triangle PQR shown in Fig. 8.22, PQ = 7 cm, PS = 3.6 cm and SQ is the altitude to the hypotenuse.

   Find the length: (a) SQ (b) SR

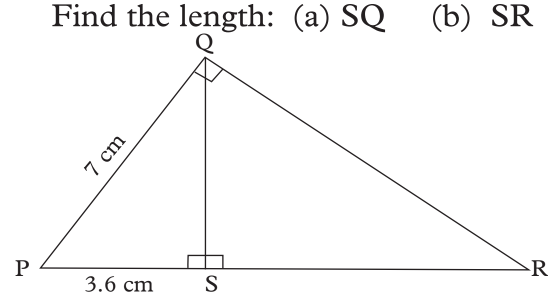


Fig 8.22

5. The altitude to the hypotenuse of a right-angled triangle is 8 cm long. If the hypotenuse is 20 cm long, what are the lengths of the two segments of the hypotenuse?

6. The altitude to the hypotenuse of a right-angled triangle divides the hypotenuse into segments that are 12 cm and 15 cm long. Find the length of the altitude.

7. Find the value of x in triangle KLM shown in Fig 8.23 given that KM = $(x + 5)$ cm, NL = x cm and NM = 6 cm.

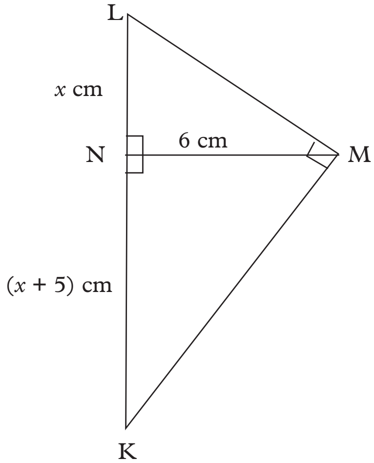


Fig 8.23

8. Triangle EFG is right angled at F. FH is the altitude to the hypotenuse and has length of 10 cm. If the lengths of the two segments EH and HG of the hypotenuse are in the ratio 1 : 4, find their actual lengths in centimetres.

   (Hint: Sketch the triangle and let EH = x)

9. The length of the hypotenuse of a right-angled triangle is 29 cm. If the length of the altitude from the hypotenuse is 10 cm long, find the lengths of the two segments of the hypotenuse.

10. The legs of a right-angled triangle are 4.5 cm and 6 cm long. Calculate the length of the altitude to the hypotenuse of the triangle.

8.4 Leg theorem of a right-angled triangle

Activity 8.6

1. Draw and cut out on a manila paper two identical right angled triangular cut outs ABC with the dimensions AC = 8 cm, CB = 6 cm AB = 10 cm and ∠ACB = 90°.
2. One of the cut out, drop a perpendicular from vetex C to meet the hypotenuse at point D.
3. Cut out the two triangles ADC and DBC and compare each of them with the 'mother' triangular cut out ABC (the represented by the unsubdivided cut out).
4. Test each of similarity with the 'mother' triangle using the corresponding angles. What do you notice?

   Measure and compare the ratios of:

   $\frac{AC\ (leg)}{AB\ (hypotenuse)}$ and

   $\frac{AD\ (projection\ of\ leg\ AC)}{AC\ (hypotenuse)}$

What do you notice?

Measure and compare the ratios;

$\frac{BC\ (leg)}{AB\ (hypotenuse)}$ and

$\frac{BD\ (projection\ of\ leg\ AC)}{BC\ (hypotenuse)}$

What do you notice?

Activity 8.7

Look at the right-angled triangles EFG and PQR in Fig 8.24 and Fig. 8.25.

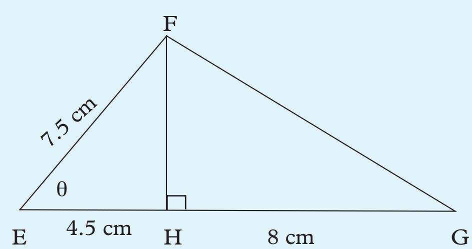


Fig 8.24

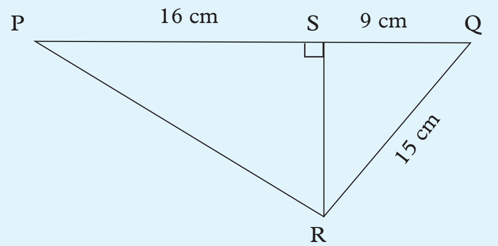


Fig 8.25

(a) Work out and compare the ratios:

(i) $\frac{EF}{EG}$ and $\frac{EH}{EF}$

(ii) $\frac{QR}{PQ}$ and $\frac{QS}{QR}$

What do you notice about each pair of ratios.

(b) Summarise your observations into a theorem on right-angled triangles.

Consider triangle UVW

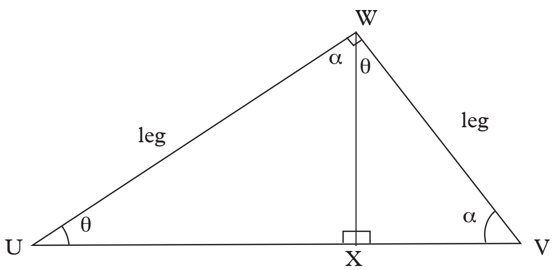


Fig. 8.26

$$\frac{UW\ (leg)}{UV\ (hypotenuse)} = \frac{UX\ (projection\ of\ leg\ UW)}{UW\ (leg)}$$

And

$$\frac{VW\ (leg)}{UV\ (hypotenuse)} = \frac{VX\ (projection\ of\ leg\ VW)}{VW\ (leg)}$$

These observations are summarized into what is known as the **leg theorem**. It states that "the leg of a right-angled triangle is the mean proportional between the hypotenuse and the projection of the leg on the hypotenuse." This is mathematically represented as;

$$\frac{leg}{hypotenuse} = \frac{projection\ of\ leg}{leg}$$

Example 8.7

Fig. 8.27 below shows a right angled triangle in which AB = 12 cm, AD = 3 cm and CD is the attitude to the hypotenuse. Find the length AC.

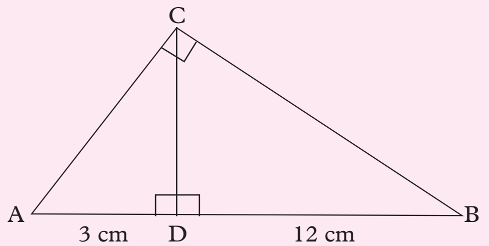


Fig. 8.27

Solution

AC is a leg, AD is the projection of AC on hypotenuse. Applying the leg theorem,

$$\frac{leg}{hypotenuse} = \frac{projection}{leg}$$

$$\Rightarrow \frac{AC}{AB} = \frac{AD}{AC}$$

Let $AC = x$

$$\frac{x}{12\ cm} = \frac{3\ cm}{x} \Rightarrow x^2 = 3\ cm \times 12\ cm$$

$$x^2 = 36\ cm^2$$

$$x = 6\ cm$$

Example 8.8

Fig. 8.28 shows a right-angled triangle PQ in which PQ = 6 cm. QT = 3.6 cm. Find the length of RT. Hence or otherwise find the length of QR.

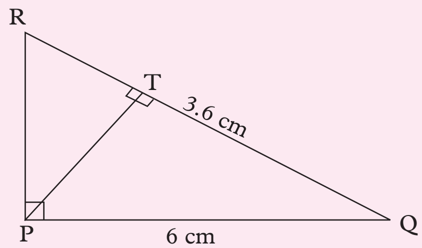


Fig. 8.28

Solution

QR is a hypotenuse of ΔPQR

PQ is a leg,

QT is the projection of leg PQ on the hypotenuse.

Applying the leg theorem

$$\frac{leg}{hypotenuse} = \frac{projection}{leg}$$

$$\frac{PQ}{QR} = \frac{QT}{PQ}$$

Let RT = x hence QR = x + 3.6 cm

$$\frac{6\ cm}{(x + 3.6)\ cm} = \frac{3.6\ cm}{6\ cm}$$

$$\Rightarrow \quad 3.6\ (x + 3.6)\ cm^2 = 36\ cm^2$$

$$x + 3.6 = \frac{3.6\ cm}{6\ cm}$$

$$\Rightarrow \quad x + 3.6 = 10$$

$$x = (10 - 3.6)\ cm = 7.4\ cm$$

$$RT = 7.4\ cm$$

$$\Rightarrow \quad QR = 7.4\ cm + 3.6\ cm = 10\ cm$$

Exercise 8.4

1. Find the length of AC in the right-angled triangle ABC shown in Fig. 8.29.

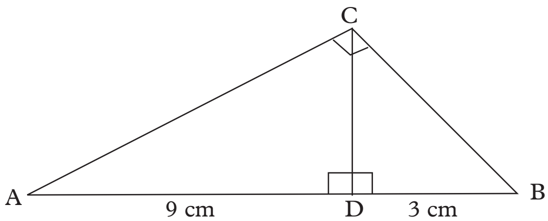


Fig. 8.29

2. What is the length PR in the right-angled triangle PQR shown in Fig. 8.30 below given that PS = 6 cm and PQ = 14 cm?

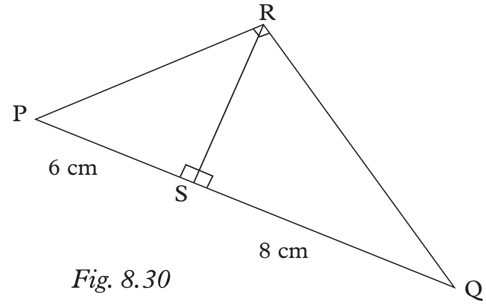


Fig. 8.30

3. Find the length NP in triangle MNO shown in Fig. 8.31 below. NO = 10 cm.

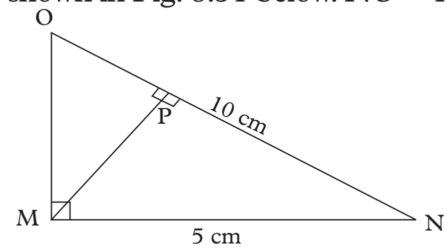


Fig. 8.31

4. What are the lengths of BD and AC in triangle ABC, Fig. 8.32, given that AD = 8 cm and CD = 12 cm?

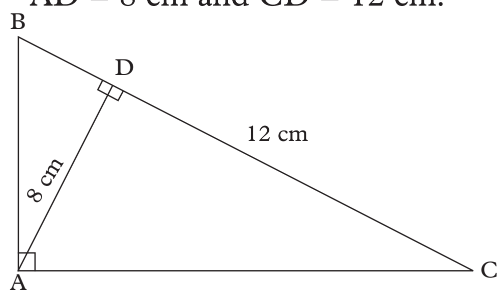


Fig. 8.32

5. Find the length of WX in triangle WXY, Fig. 8.33, given that WY = 7 cm and YB = 9 cm.

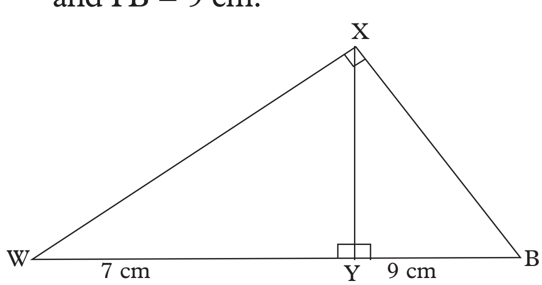


Fig. 8.33

6. The hypotenuse of a right-angled triangle is 5 cm long and the longer leg of the triangle is 4 cm long. What is the length of the projection of the shorter leg on the hypotenuse?
7. In triangle UVW in Fig. 8.34, UX = 6 cm and VW = 8 cm.

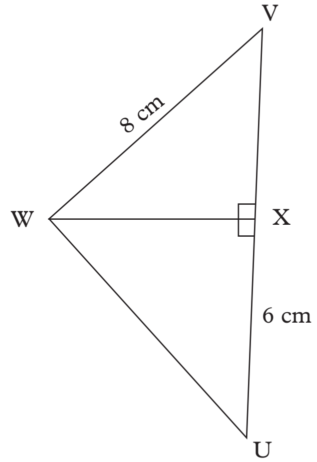


Fig. 8.34

Find: (a) XV (b) XW

8. Fig. 8.35 below shows triangle KLM. Given that KM = 12 cm, ML = 9cm and KL = 15 cm, find;

   (a) KN (b) NL (c) NM

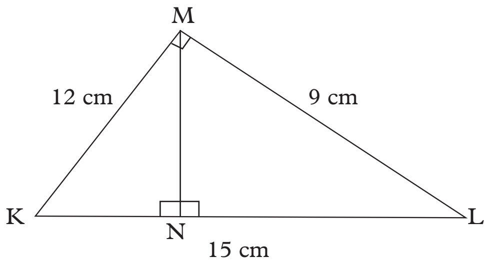


Fig. 8.35

9. In Fig. 8.36, O is the centre of the circle. PR = 12 cm and PT = TU = UR.

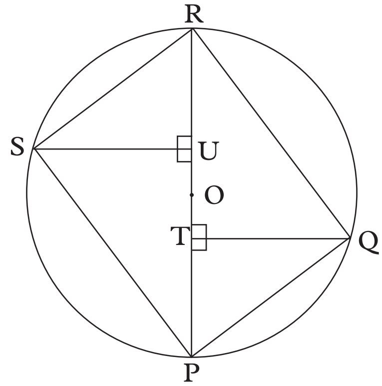


Fig. 8.36

Find; (a) QR (b) PQ (c) SU

10. The altitude to the hypotenuse of a right-angled triangle EFG divides the hypotenuse into 9 cm and 12 cm segments.

    Find the lengths of the:

    (a) altitude to the hypotenuse
    (b) shorter leg of triangle EFG
    (c) longer leg of triangle EFG.

8.5 Introduction to trigonometry

Activity 8.8

With reference to angle θ, name the adjacent side, the opposite side and the hypotenuse in each of the triangles in Fig. 8.37.

(a)

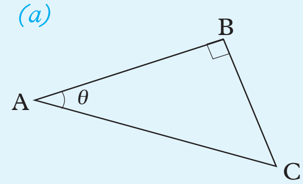


Fig. 8.37

(b)

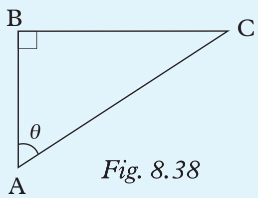


Fig. 8.38

(c)

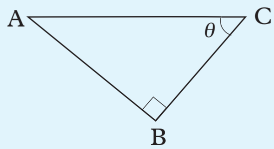


Fig. 8.39

The word '**trigonometry**' is derived from two Greek word: *trigonon* meaning a triangle and *metron*, meaning measurement. Thus trigonometry is the branch of mathematics concerned with the relationships between the sides and the angles of triangles.

In trigonometry, Greek letters are used to indicate, in a general way, the sizes of various angles.

The most commonly used of these letters are:

α = alpha; β = beta; γ = gamma;
δ = delta; θ = theta; ϕ = phi;
ω = omega.

In senior 2, we studied the relationship between the lengths of the sides of a right-angled triangle, which is known as the Pythagorean relationship. In this unit, we shall study the relationships between the acute angles and the sides of a right-angled triangle.

The sides of such a triangle are named, with reference to specific angle e.g. θ, as shown in Fig. 8.40.

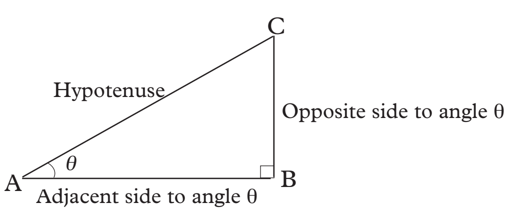


Fig. 8.40

If C is used as the reference angle in Fig 8.41 AB becomes the opposite side and BC the adjacent side. These definitions of the sides apply to any right-angled triangle in any position.

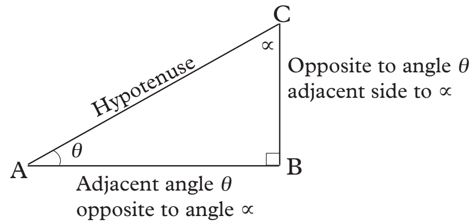


Fig. 8.41

Example 8.9

Consider triangle PQR in Fig. 8.42.

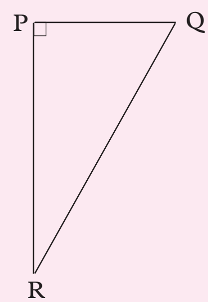


Fig. 8.42

Identify:

(a) the hypotenuse

(b) identify the:

(i) side opposite to ∠R

(ii) side adjacent to ∠R

` *(iii) side opposite to ∠Q*

(iv) side adjacent to ∠Q

Solution

(a) hypotenuse is QR

(b) (i) PQ (ii) PR (iii) PR
(iv) PQ

Exercise 8.5

1. Name the adjacent side, the opposite side and the hypotenuse relative to each of the marked angles in Fig. 8.44.

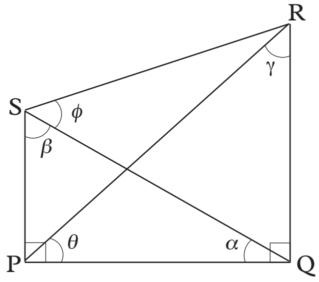


Fig. 8.44

2. Using Fig. 8.45, in which all the lengths are in centimetres, find the length of the side which is:
   (a) opposite to ∠BAC
   (b) adjacent to ∠ADC.

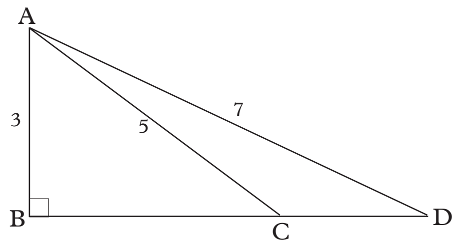


Fig. 8.45

3. Using Fig. 8.46, in which all lengths are in centimetres, name and give the lengths of two sides which are:
   (a) adjacent to angle θ,
   (b) hypotenuse relative to angle α.

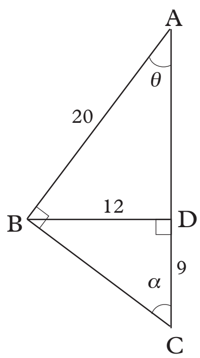


Fig. 8.46

8.6 Trigonometric Ratios

In any right-angled triangle, there are three basic ratios between the sides of the triangle with reference to a particular acute angle in the triangle. They are sine, cosine and tangent.

8.6.1 Sine and cosine of an acute angle

Activity 8.9

1. Construct ΔAOB such that ∠AOB = 30°, ∠ABO = 90° and OB = 6 cm.
2. On line OB mark points B_1, B_2, B_3, B_4 and B_5 such that $OB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5 = B_5B = 1$ cm. Through the points, draw lines parallel to AB meeting line OA at A_1, A_2, A_3, A_4 and A_5 respectively.
3. Measure A_iB_i (i.e. A_1B_1, A_2B_2, …).
4. Measure OA_i (i.e. OA_1, OA_2 …).

Copy and complete Table 8.2.

i	1	2	3	4	5
OA_i (cm)					
OB_i (cm)	1	2	3	4	5
A_iB_i (cm)					
$\frac{A_iB_i}{OA_i}$					
$\frac{OB_i}{OA_i}$					

Table 8.1

5. In each case, what do you notice about the ratios $\frac{AiBi}{OAi}$ and $\frac{OBi}{OAi}$?

Consider a right-angled triangle ABC in Fig. 8.47 below.

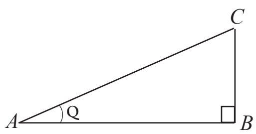


Fig. 8.47

For ∠A, BC is opposite, AB is adjacent and AC is the hypotenuse.

The ratio $\frac{BC}{AC} = \frac{\text{opposite side to } \angle\theta}{\text{hypotenuse}}$ is the sine of angle θ.

This ratio is denoted as; $\frac{BC}{AC}$ = sine of θ, written in short as

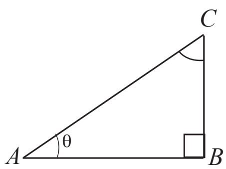


Fig. 8.48

$$\text{Sin } \theta = \frac{BC}{AC} = \frac{\text{Opposite side to } \angle A}{\text{Hypotenuse}}$$

Consider ΔABC in Fig. 8.49;

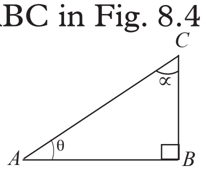


Fig. 8.49

Similarly, $\frac{AB}{AC} = \frac{\text{Opposite side to } \angle\propto}{\text{Hypotenuse}}$

is the sine of angle ∝

$$\therefore \quad \text{Sin } \propto = \frac{AB}{AC} = \frac{\text{opposite side to } \angle C}{\text{hypotenuse}}$$

The sine of any of the two angles is equal to

$$\frac{\text{opposite side to the given angle}}{\text{hypotenuse}}$$

Consider ΔABC in Fig. 8.50

The cosine of angle θ is defined as the ratio of the $\frac{\text{adjacent side to angle } \theta}{\text{hypotenuse}}$

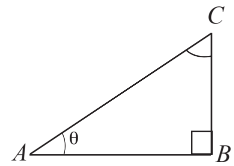


Fig. 8.50

$$\text{Thus, } \cos\theta = \frac{\text{adjacent side to angle } \theta}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\text{Also, } \cos\propto = \frac{\text{opposite side to } \angle\propto}{\text{hypotenuse}}$$

$$= \frac{BC}{AC}$$

In general, given a right-angled triangle ABC as shown in Fig. 8.51.

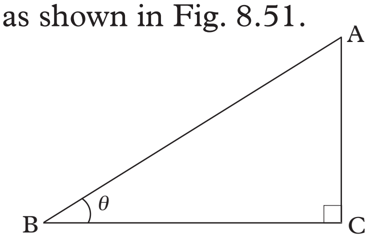


Fig. 8.51

$$\text{Sin } \theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{AC}{AB}, \text{ and}$$

$$\text{Cos } \theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{BC}{AB}$$

Example 8.10

Consider the triangle ABC with AB = 1 cm, AC = 1 cm.

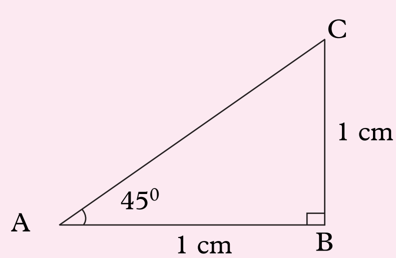


Fig. 8.52

(a) Find the length of side AC. Leave result in surds

(b) Find from the triangle:

(i) Cos 45

(ii) Sin 45

Leave the answer in surds.

Solution

(a) By Pythagoras theorem;

$AC^2 = AB^2 + BC^2$

$AC^2 = 1^2 + 1^2$

$AC^2 = 2$

$AC = \sqrt{2}$ cm

(b) (i) $\text{Cos } 45 = \frac{AB}{AC} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

(ii) $\text{Sin } 45 = \frac{BC}{AC} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Exercise 8.6

1. By drawing and measuring, find the approximate values for:
   (a) sin 20°, cos 20°
   (b) sin 42°, cos 42°
   (c) sin 65°, cos 65°
   (d) sin 78°, cos 78°
2. Find, by drawing and measuring, approximate sizes of the angles whose sines and cosines are given below.
   (a) $\sin A = \frac{2}{3}$ (b) $\sin B = \frac{3}{4}$
   (c) $\sin C = \frac{3}{5}$ (d) $\cos D = \frac{1}{3}$
   (e) $\cos E = \frac{3}{7}$ (f) $\cos F = \frac{5}{7}$
   (g) sin G = 0.42
   (h) sin H = 0.84
   (i) sin I = 0.65
   (j) cos J = 0.23
   (k) cos K = 0.34
   (l) cos L = 0.56

   Hint for parts (g) to (l): First convert the decimals into fractions.

8.6.2.1 Finding sine and cosine using calculators

Activity 8.10

Use your scientific calculator to find the sines and cosines of the following.

(a) 32° (b) 17.89° (c) 73.5°

Note that as the angles increase from 0° to 90°,

(i) their sines **increase** from 0 to 1, and

(ii) their cosines **decrease** from 1 to 0.

Example 8.11

Use calculators to find the value of:

(a) sin 53.4° (b) cos 71.2°

Solution

(a) Press sin, type 53.4° press =
(0.802817… is displayed)
Thus, sin 53.4° = 0.802 8

(b) Press cos, type 71.2° press =
(0.3222656… is displayed)
Thus, cos 71.2° = 0.322 3

Example 8.12

Use a calculator to find the angle whose

(a) *sine is 0.866*

(b) *cosine is 0.7071*

Solution

(a) *Let θ be the angle* whose sine is 0.866.

So, Sin $\theta = 0.866$.

We use sin^{-1} *function called Sine inverse to find the value of angle* θ

$\therefore \ \theta = sin^{-1}(0.866) = 60^{o}$.

On your scientific calculator, press shift, press sine button, type 0.866, press equal signs.

We get $\theta = 60°$

(b) *Let* $\propto$ *be the angle whose cosine is 0.7071*

So, Cos $\propto = 0.7071$

We use cos^{-1} *function called Cosine inverse to find the value of angle* $\propto$

$\therefore \ \theta = cos^{-1}(0.7071) = 45^{o}$

On your scientific calculator, press shift, press cosine button, type 0.7071, press equal signs.

We get $\theta = 45°$

Exercise 8.7

1. Use calculators to find the sine of:

(a)	3°	(b)	13°	(c)	70°
(d)	63°	(e)	13.2°	(f)	47.8°
(g)	79.2°	(h)	89.2°		

2. Use calculators to find the cosine of:

(a)	18°	(b)	27°	(c)	49°
(d)	70°	(e)	19.5°	(f)	36.6°
(g)	77.7°	(h)	83.9°		

3. Use calculators to find the angle whose sine is:

(a)	0.397 1	(b)	0.788 0
(c)	0.927 8	(d)	0.996 3
(e)	0.948 9	(f)	0.917 8

4. Use calculators to find the angle whose cosine is:

(a)	0.918 2	(b)	0.564 1
(c)	0.123 4	(d)	0.432 1
(e)	0.880 1	(f)	0.555 5

8.6.2.2 Using sines and cosines to find angles and lengths of sides of right-angled triangles

Activity 8.11

Use your knowledge of sines and cosines to find the values of unknown sides and angles in the following triangles.

(a) (b)

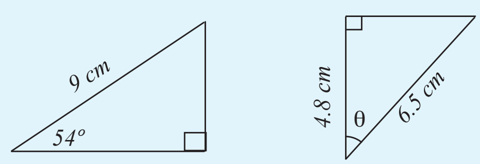


Fig. 8.53

Example 8.13

Find the value of x and y in Fig. 8.54 to 2 s.f.

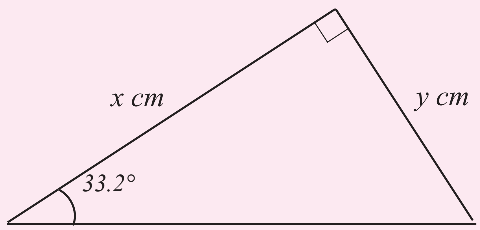


Fig. 8.54

Solution

$\sin 33.2^o = \frac{\text{opposite side}}{\text{hypotenuse}}$

$\sin 33.2° = \frac{y}{8}$

$\Rightarrow y = 8 \sin 33.2° = 8 \times 0.547\ 6$ cm

$= 4.380\ 8$ cm

$= 4.4$ cm (2 s.f.)

$\cos 33.2^o = \frac{\text{adjacent side}}{\text{hypotenuse}}$

$\cos 33.2° = \frac{x}{8}$

$\Rightarrow x = 8 \cos 33.2° = 8 \times 0.836\ 8$ cm

$= 6.694\ 4$ cm

$= 6.7$ cm (2 s.f.)

Example 8.14

In a quadrilateral ABCD, AB = 8 cm, AC = 10 cm and CD = AD = 7 cm and ∠B = 90°

Calculate:

(a) ∠BAC (b) ∠ADC (c) side BC

Solution

Draw a sketch of the quadrilateral with all the given information (Fig. 8.55).

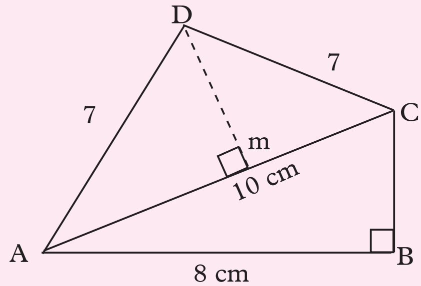


Fig. 8.55

(a) Using ΔABC,

$\text{Cos } A = \frac{AB}{AC} = \frac{8}{10} = 0.8$

To get ∠A, we need to get $\cos^{-1} 0.8$ in a calculator,

press $\cos^{-1}$, type 0.8 and press =

we get $\cos^{-1} 0.8 = 36.9°$

Thus ∠ A = 36.9°

$\therefore \angle A = 36.9°$

(b) Using ΔADC, drop a perpendicular to AC so as to create right-angled. Let the perpendicular from D meet AC at M. Since ΔADC is isosceless, the perpendicular line from D bisects AC.

∴ Using ΔADM,

$\sin D = \frac{MA}{DA}$

$= \frac{5}{7}$

$= 0.71428571$

Sin D = 0.7143

To get ∠D, we need to get sin-1 0.7143 in a calculator,

press $\sin^{-1}$, type 0.7143 then press=

We get $Sin^{-1}\ 0.7143 = 45.6^o$

Therefore ∠D = 45.6°

∴ ∠ADC = 2∠ADM (DM bisects angle ADC)

$= 2 \times 45.6^o$

$= 91.2°$

(c) Using ΔABC, or using Pythagoras theorem,

$Sin\ A = \frac{BC}{AC}$	or $AB^2 + BC^2 = AC^2$
$BC = AC\ sin\ A$	$8^2 + BC^2 = 10^2$
$= 10 \times 0.6008$	$BC^2 = 100 - 64$
$= 6.008$	$= 36$
$BC = 6.008cm$	$BC = \sqrt{36}$
	$= 6$

$\therefore BC = 6$ *cm long.*

Note: *The discrepancy in the two answers is due to the use of approximation in the first method.*

Exercise 8.8

1. In Fig.8.56, find the lengths marked with letters. The lengths are in centimetres. Give your answer correct to 3 decimal places.

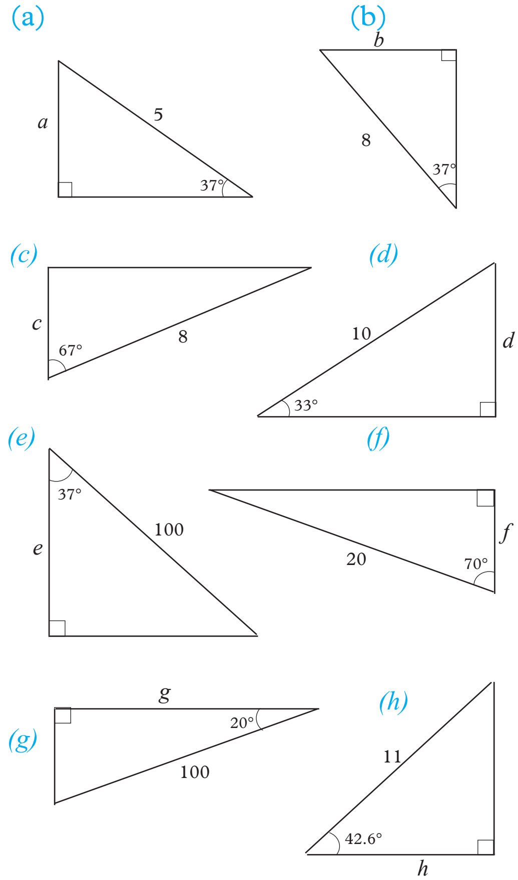


Fig. 8.56

2. Find the lengths marked with letters in Fig. 8.57. Measurements are in centimetres. Give your answers to 3 significant figures.

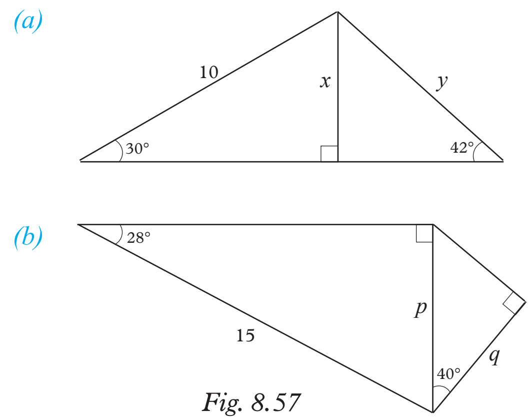


Fig. 8.57

8.6.2 Tangent of an acute angle

8.6.2.1 Defination of tangent of an angle

Activity 8.12

1. Draw $\angle AOB = 35°$.
2. On OB, mark points B_1, B_2, B_3, B_4, and B_5 such that $OB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5 = 1$ cm.
3. Construct perpendicular lines through B_1, B_2, B_3, B_4 and B_5 to cut OA at A_1, A_2, A_3, A_4 and A_5 respectively.
4. Measure A_1B_1, A_2B_2, A_3B_3, A_4B_4 and A_5B_5.
5. Copy and complete Table 8.2, where A_iB_i means A_1B_1, A_2B_2, etc.

i	1	2	3	4	5
A_iB_i					
OB_i	1 cm	2 cm	3 cm	4 cm	5 cm
$\frac{A_iB_i}{OB_i}$					

Table 8.2

6. What do you notice about the ratio $\frac{AiBi}{OB}$?

Consider ΔABC in Fig. 8.58, and ∠A

The ratio $\frac{BC}{AB} = \frac{\text{Opposite to } \angle A}{\text{Adjacent to } \angle A}$ is called the tangent of the angle at A.

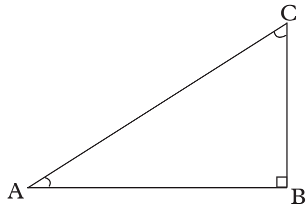


Fig. 8.58

This ratio is denoted as $\frac{BC}{AB}$ = tangent of θ written in short as $\tan\theta = \frac{BC}{AB}$.

Similarly, with reference to the angle at C in Fig. 8.59, $\frac{AB}{BC} = \frac{\text{opposite to } \angle C}{\text{adjacent to } \angle C}$ is called the tangent of angle C.

$\text{Tan} \propto = \frac{AB}{BC}$.

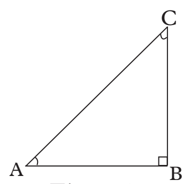


Fig. 8.59

In general, given a right-angled triangle ABC (Fig. 8.60)

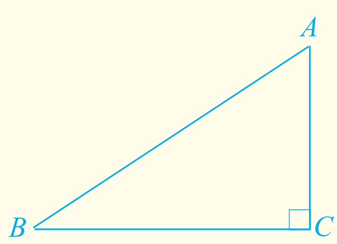


Fig. 8.60

The ratio $\frac{AC}{OB}$ is called the tangent of ∠B.

Similarly, $\frac{BC}{AC}$ is the tangent of ∠A.

In short, we write

$\tan B = \frac{AC}{BC}$ and $\tan A = \frac{BC}{AC}$

Thus $\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$, where θ is one adjacent side of the acute angles in a right-angled triangle.

Example 8.15

By drawing and measuring, find the value of tan 30°.

Solution

Draw an angle AOB = 30°

On OB, mark off OP = 5 cm

At P draw a perpendicular line to cut OA at Q (Fig. 8.61).

Measure PQ.

$$\tan 30° = \frac{PQ}{OP} = \frac{2.9}{5} = 0.58$$

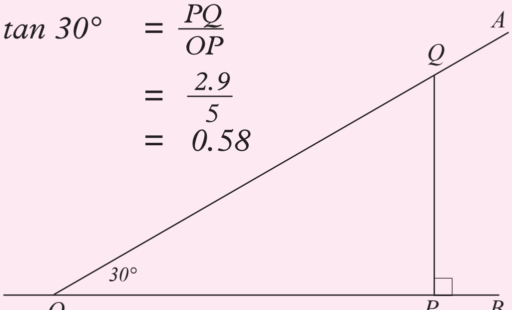


Fig. 8.61

Example 8.16

By drawing and measuring, find the angle whose tangent is $\frac{2}{3}$.

Solution

Let the angle be θ.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{3}$$

The lengths of the opposite and adjacent sides are in the ratio 2 : 3.

So we can use 2 and 3 cm.

Draw AB = 3 cm. At B, draw BC = 2 cm perpendicular to AB. Join AC (Fig. 8.62)

Measure ∠CAB.

By measurement, ∠CAB = 34°.

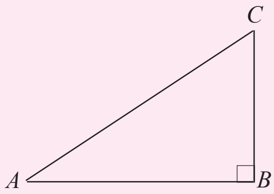


Fig 8.62

Exercise 8.9

1. Find the tangents of the following angles by drawing and measuring.

(a) 40° (b) 45° (c) 50°

(d) 25° (e) 33° (f) 60°

2. By drawing and measuring, find the angles whose tangents are as follows:

(a) $\frac{1}{4}$ (b) $\frac{2}{5}$ (c) $\frac{3}{7}$

(d) $\frac{5}{4}$ (e) $\frac{6}{5}$ (f) $\frac{7}{4}$

(g) $\frac{8}{3}$ (h) 1 (i) 3

8.6.2.2 Use calculators to find tangent of angles and angles given their tangents

Activity 8.13

Use scientific calculator to find tangent of:

(a) 7° (b) 13.6° (c) 50°

In lower levels, we learnt how to use calculators in simple mathematical operations. We can also use calculators to find tangents, cosines or sines of given angles. The following examples will show how to find tangents of given angles.

Example 8.17

Use calculators to find the tangents of the following:

(a) 67° (b) 60.55° (c) 38.88°.

Solution

(a) (Press tan, type 67 and press) =
tan 67° = 2.355 9

(b) To find tan of 60.55°:
Press tan enter 60.55
Press =, (1.7711 is displayed)
∴ tan 60.55° = 1.771 1

(c) tan 38.88°
Press tan, type 38.88
Press = (0.8063 is displayed
∴ tan 38.88° = 0.806 3

Example 8.18

Use a calculator to find the angle whose tangent is:

(a) 1.28 (b) 0.875

Solution

We use the function tan^{-1} read as inverse to find the angle whose tangent is given.

(a) tan^{-1} 1.28
Press tan^{-1}, type 1.28 then press =
We get tan^{-1} 1.28= 52°
The angle is 52°.

(b) tan^{-1} 0.875
Press tan^{-1}, type 0.875 then press =
We get tan^{-1} 0.875= 41.1°
The angle is 41.1°.

Exercise 8.10

Use a calculator to find the tangents of the following angles:

(a)	7°	(b)	13.6°	(c)	50°
(d)	47.7°	(e)	80.5°	(f)	2.21°
(g)	18.46°	(h)	55.68°	(i)	66.99°
(j)	51°	(k)	64°	(l)	73°

8.6.2.3 Using tangents to find lengths of sides and angles of right-angled triangles

Activity 8.14

1. Use the knowledge you have gained so far to find the sides and the angles marked with letters in the following triangles.

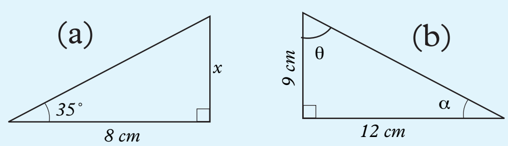


Fig. 8.63

2. Using the expression for the tangent of acute angles in a right angled triangle, we are able to find the values unknown sides angles in the triangle as demonstrated in the following examples.

Example 8.19

Find the length indicated x in Fig. 8.64.

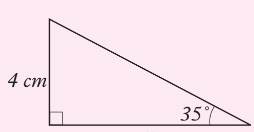


Fig. 8.64

Solution

The two indicated sides are the opposite and the adjacent sides.

$\therefore \tan 35^\circ = \frac{4}{x}$

$\Rightarrow x = \frac{4}{\tan 35^\circ}$

$= \frac{4}{0.7002} = 4 \times \frac{1}{0.7002}$

$= 4 \times 1.429$

$\therefore \quad x \approx 5.7$ *cm (2 s.f)*

Example 8.20

ABC is a triangle such that AB = 8cm, ∠B = 90° and BC = 6 cm. M is the midpoint of BC. Calculate:

(a) ∠BAM (b) ∠MAC

Solution

Draw a sketch of the triangle ABC, showing all the given information, as in Figure 8.65.

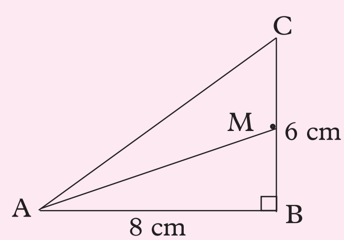


Fig. 8.65

(a) *Join A to M. Using ΔABM, BM = 3cm …. (midpoint of BC)*

$\tan B\hat{A}M = \frac{BM}{AB} = \frac{3}{8} = 0.375$

$\tan^{-1} 0.375 = \angle BAM$

In a calculator, press tan⁻¹ type 0.75 and press =

$\therefore \quad \angle BAM = 20.5^\circ$

$= \frac{BM}{AB} = \frac{3}{8} = 0.375$

(b) *Using* ΔABC*, we can find* $\angle BAC$

$\therefore \tan \angle BAC = \frac{BC}{AB} = \frac{6}{8} = 0.75$

$\tan^{-1} 0.75 = \angle BAC$

In a calculator, press $\tan^{-1}$*, type 0.75 and press =*

$\angle BAC = 36.5°$

$\angle MAC$ *is the between* $\angle BAC$ *and*

$\angle BAM$

$\therefore \angle CAM = \angle BAC - BAM$

$= 36.5° - 20.5°$

$= 16°$

$\therefore \angle CAM = 16°$

Exercise 8.11

1. Find the length of the side indicated in each of the triangles in Fig. 8.66. State your answer to 3 s.f.

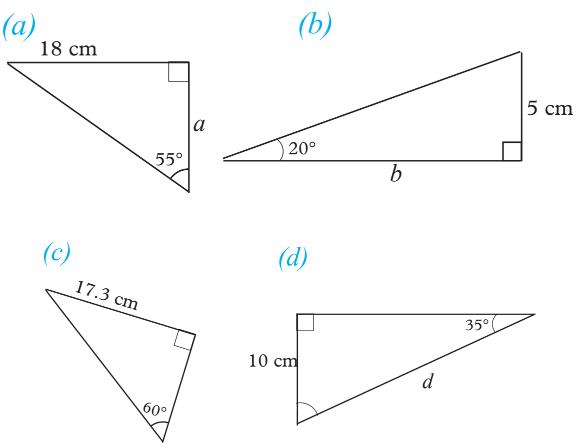
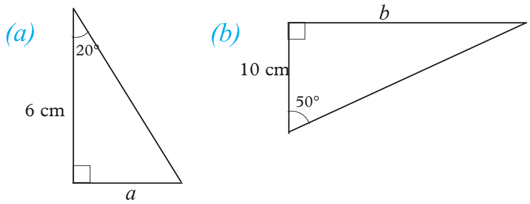


Fig. 8.66

2. Find the length of the side indicated in each of the triangles in Fig 8.67. State your answer in 3 s.f.



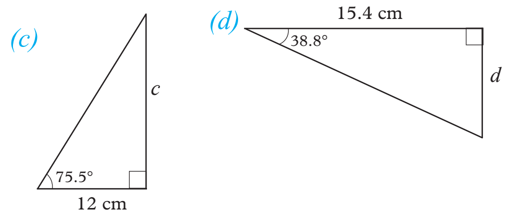


Fig. 8.67

3. For this question, use ΔABC which is right angled at B to calculate the required angle given the following information;
   (a) AB = 8 cm, BC = 4.25 cm. Calculate angle BAC.
   (b) AB = 12cm, BC = 5 cm. Calculate ∠A.
   (c) AC = 6 cm, BC = 2.82 cm. Calculate ∠A.
   (d) AC = 9 cm, AB = 5.03 cm. Calculate ∠BAC.
   (e) AC = 15 cm BC = 11 cm, calculate ∠ACB.

8.6.3 Application of trigonometic ratios (sine, cosine and tangent)

Activity 8.15

Let A be the foot of the tower and x be the height of the tower which can be viewed from points B and C at angles α and β as Fig 8.68 below shows. Points B and C are a cm apart.

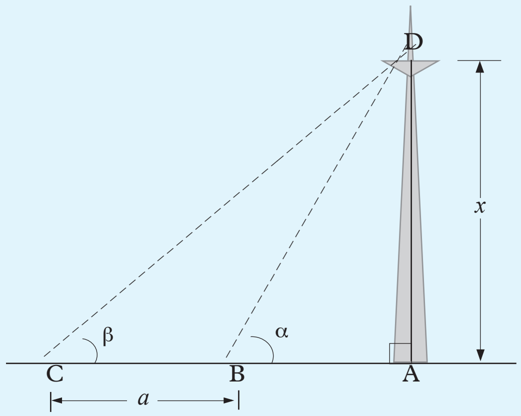


Fig. 8.68

Obtain an expression for *a* in terms of x, β and α.

The following examples illustrate some of the calculations involved when faced with real life situations.

Example 8.21

A hawk is perched on a tree at a height(H) of 15 m above the ground. It spots a chicken, sitting on the ground, at an angle of depression of 25° (Fig. 8.69). How far from the tree is the chicken?

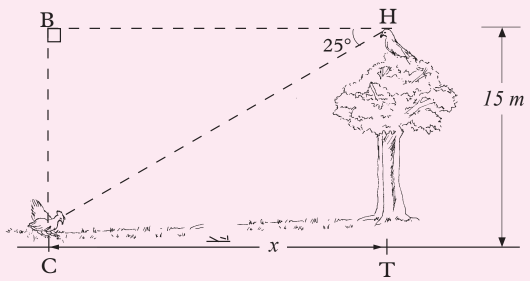


Fig. 8.69

Solution

Let the distance from the tree T to the chicken C be x m.

To find x, we use any of the following two methods.

Method 1

$\angle HCT = \angle BHC = 25°$ *(alternate angles) and* ΔCHT *is right-angled.*

$\therefore$ Using ΔHCT, $\tan 25° = \dfrac{15}{x}$

$\Rightarrow \quad x \tan 25° = 15$

$\Rightarrow \quad x = \dfrac{15}{\tan 25°}$

.

Method 2

In ΔHCT, $\angle CTH = 90°$

$\therefore \angle CHT = 180° - (90° + 25°)$

$= 65°$

Now, $\tan 65° = \dfrac{x}{15}$

$\Rightarrow \quad x = 15 \tan 65°$

$= 32.17$ m

Note that 65° is the **complement** of 25°, and that it is easier to multiply 15 by tan 65° (Method 2) than to divide 15 by tan 25° (Method 1). Hence, given a problem like the one above, it is better to find the **complement** of the given angle and then use it to solve the problem. However, if you are using a calculator, either method will do.

Example 8.22

Two observers, A and B, 500 m apart, observe a kite in the same vertical plane and from the same side of it. The angles of elevation of the kite are 20° and 30° respectively. Find the height of the kite, disregarding the height of the observers.

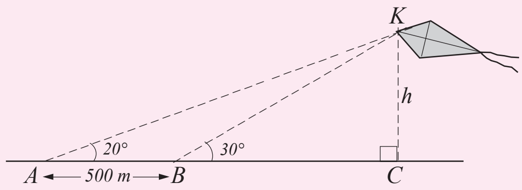


Fig. 8.70

Solution 1

Fig. 8.70 shows the kite and the relative positions of A and B.

From Fig. 8.81,

$$\tan 20° = \frac{h}{AC}$$

$$\therefore\ AC = \frac{x}{\tan 20^o}$$

$$\text{Also } \tan 30° = \frac{h}{BC}$$

$$\therefore\ BC = \frac{x}{\tan 30^o}$$

$$\text{Now, } AC - BC = \frac{h}{\tan 20^o} - \frac{h}{\tan 30^o}$$

$$\text{i.e. } AC - BC = h(\frac{h}{\tan 20^o} - \frac{h}{\tan 30^o})$$

$$\Rightarrow\ 500 = h(\frac{1}{0.364\,0} - \frac{1}{0.577\,4})$$

$$500 = h\,(2.747 - 1.732)$$

$$500 = 1.015\,h$$

$$\therefore\ h = \frac{500}{1.015} \approx 492.6\text{ m.}$$

Exercise 8.12

1. A ladder of length 5.5 m rests against a vertical wall so that the angle between the ladder and the ground is 60°. How far from the wall is the foot of the ladder?
2. A boy is flying a kite using a string of length 56 m. If the string is taut and it makes an angle of 62° with the horizontal, how high is the kite? (Ignore the height of the boy).
3. Find the dimensions of the floor of a rectangular hall given that the angle between a diagonal and the longer side is 25° and that the length of the diagonal is 10 m.
4. A plane takes off from an airport and after a while, an observer at the top of the control tower sees it at an angle of elevation of 9°.

   At that instant, the pilot reports that he has attained an altitude of 2.4 km. If the height of the control tower is 50 m, find the horizontal distance that the plane has flown?
5. After walking 100 m up a sloping road, a man finds that he has risen 30 m. What is the angle of slope of the road?
6. The tops of two vertical poles of heights 20 m and 15 m are joined by a taut wire 12 m long. What is the angle of slope of the wire?
7. A man walks 1 000 m on a bearing of 025° and then 800 m on a bearing of 035°. How far north is he from the starting point?
8. Two boats A and B left a holiday resort at the coast. Boat A travelled 4 km on a bearing of 030° and boat B travelled 6 km on a bearing of 130°.
   (a) Find which boat travelled further eastwards and by how much.
   (b) How far to the north is boat A from boat B?
9. A bridge crosses a river at an angle of 60°. If the length of the bridge is 170 m, what is the width of the river?
10. A man sitting at a window with his eye 20 m above the ground just sees the sun over the top of a roof 45 m high. If that roof is 30 m away from him horizontally, find the angle of elevation of the sun.
11. The shaft of a mine descends for 100 m at an angle of 13° to the horizontal and then for 200 m at an angle of 7° to the horizontal. How far below the starting point is the end of the shaft?

12. Two girls, one east and the other west of a tower, measure the angles of elevation of the top its spire as 28° and 37°. If the top of the spire is 120 m high, how far apart are the girls?
13. Jacob and Bernard stand on one side of a tower and in a straight line with the tower. They each use a clinometer and determine the angle of elevation of the top the tower as 30° and 60° respectively. If their distance apart is 100 m, find the height of the tower.
14. Fig. 8.71 shows the side view of an ironing board. The legs are all 95 cm long and make 60° with the floor when completely stretched.
    (a) How high is the ironing surface from the floor, given that the board is 2.5 cm thick?
    (b) How far apart are the legs at the floor?

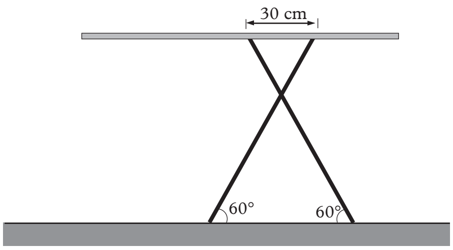


Fig.8.71

Unit Summary

- **Pythagoras theorem**

  Consider the right-angled triangle ABC shown in Fig 8.72.

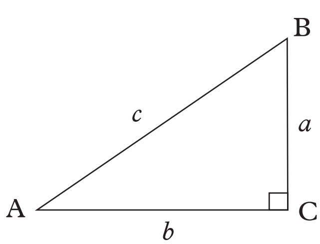


Fig. 8.72

Pythagoras theorem states that

$a^2 + b^2 = c^2$

- **The median theorem** of a right-angled triangle states that.

  The median from the right-angled vertex to the hypotenuse is half the length of the hypotenuse.

  Consider the right angled triangle XYZ Fig. 8.73.

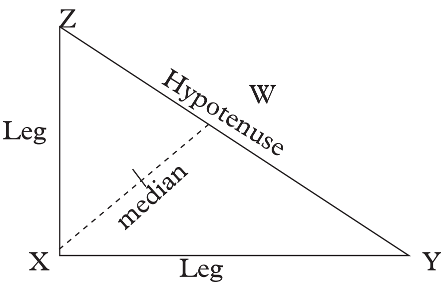


Fig. 8.73

$WX = \frac{1}{2}YZ$

Hence WX = WZ = WY

- The **altitude theorem** of a right-angled triangle states that:

  "The altitude to the hypotenuse of a right-angled triangle is the mean proportional between the segments into which it divides the hypotenuse."

  Consider the right-angled triangle EFG. Fig. 8.74.

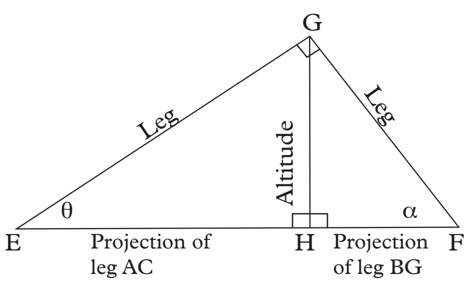


Fig. 8.74

$$\frac{\textit{altitude}}{\textit{part of hypotenuse}} = \frac{\textit{other part of hypotenuse}}{\textit{altitude}}$$

$$\frac{HG}{EH} = \frac{HF}{HG}$$

- The **leg theorem** of a right-angled triangle states that "the leg of a right-angled triangle is the mean proportional between the hypotenuse and the projection of the leg on the hypotenuse." It can be simply presented as;

$$\frac{\textit{leg}}{\textit{hypotenuse}} = \frac{\textit{projection of leg}}{\textit{leg}}$$

Consider the right-angled triangle UVW.

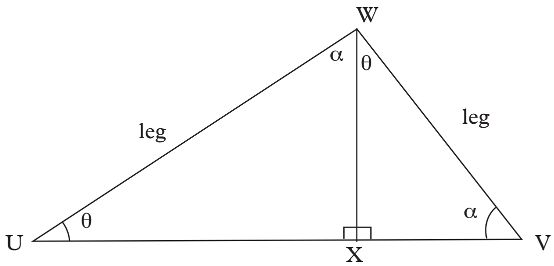


Fig. 8.75

$$\frac{\textit{UW (leg)}}{\textit{UV (hypotenuse)}} = \frac{\textit{UX (projection of leg UW)}}{\textit{UW (leg)}}$$

And

$$\frac{\textit{VW (leg)}}{\textit{UV (hypotenuse)}} = \frac{\textit{VX (projection of leg VW)}}{\textit{VW (leg)}}$$

- Naming of sides and angles in a right-angled triangle

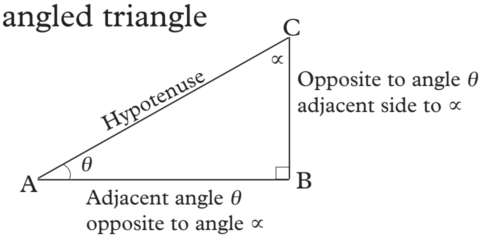


Fig. 8.76

- Tangent of an acute angle

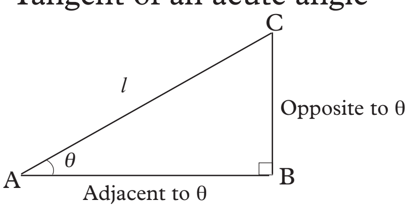


Fig. 8.77

$$\text{Tan } \theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{BC}{AB}$$

- Sine of an acute angle

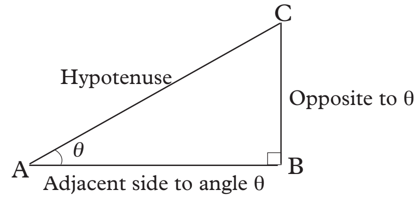


Fig. 8.78

$$\text{Sin } \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{BC}{AC}$$

- Cosine of an acute angle.

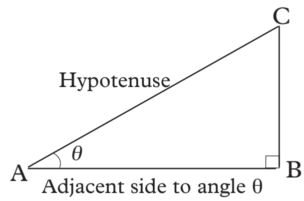


Fig. 8.79

$$\text{Cos } \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{AB}{AC}$$

Unit 8 Test

1. Find the missing sides in Fig. 8.80. (Measurements are in cm).

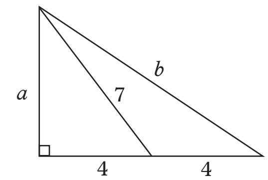


Fig. 8.80

2. A ladder which is 6 m long leans against a wall. If the top of the ladder is 4 m above the ground, how far from the wall is the foot of the ladder?
3. In a right-angled triangle, the median to the hypotenuse is 6.4 cm. What is the length of the hypotenuse.
4. In a right-angled triangle the length of the median to the hypotenuse is

(3x – 7) cm long. The hypotenuse is (5x – 4) cm long. Find the length of the hypotenuse.

5. Find the lengths of the sides marked with letters in the following triangle.

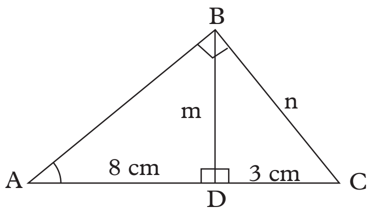


Fig. 8.81

6. A right-angled triangle ABC has its leg a = 5 cm long and altitude to the hypotenuse h = 3 cm. Find the length of sides b and c.

7. In a right-angled triangle the altitude to the hypotenuse is 8 cm high. The hypotenuse is 20 cm long. Find the lengths of the segements of the hypotenuse (Hint: let the length of the segment be x.)

8. Find the length AB in the triangle ABC below Fig. 8.82.

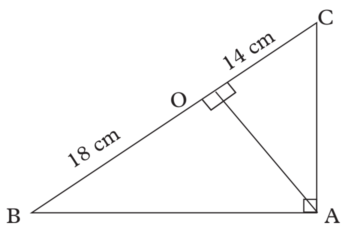


Fig. 8.82

9. A boat is 300 m from a vertical cliff. The angle of elevation of the top of the cliff is 30°. After the boat moves a distance x metres towards the cliff, the angle of elevation becomes 70°. Find the value of x, to the nearest 1 m.

10. Two tall buildings A and B are 40 m apart. From foot A, the angle of elevation of the top of B is 60°. From the top of A, the angle of depression of the top of B is 30°. Find the heights of A and B, to the nearest 1 m.

11. Find the value of the side marked with letters in the following figures:

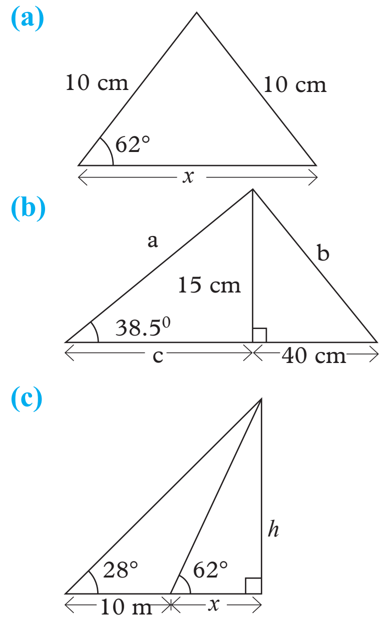


Fig. 8.83

9 CIRCLE THEOREM

Key unit competence

By the end of the unit, learners should be able to construct mathematical arguments about circles and discs. Use circle theorems to solve related problems.

Unit outline

- Elements of circles and discs.
- Circle theorem.

Introduction

Unit Focus Activity

1. Research from reference books or Internet the angle theorems for circles.
2. Apply the theorems to find the values of angles given in Fig. 9.1 below given that K, L, M and N are points on the circumference of a circle centre O. The points K, O, M and P are on a straight line.

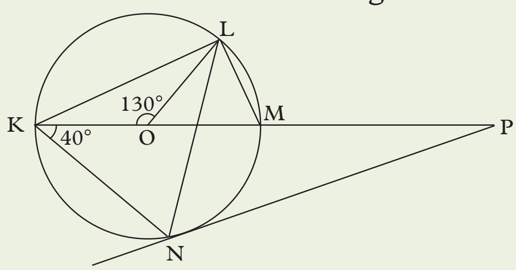


Fig 9.1

(a) ∠MLN (b) ∠OLN
(c) ∠LNP (d) ∠MPN

3. State the reasons and theorem supporting for your in each case.
4. Compare your solutions with those of other classmates in a class discussion.

In P5, we already learned some basics concepts about circles, their properties and how to get their area and circumference.

We also learnt types of angles and he angle sum in a triangle. In this unit, we will investigate the relationships between angles when they are drawn in a circle. Let us begin this by refreshing our knowledge on elements of a cricle.

9.1 Elements of a circle and disk

Activity 9.1

1. Use a pair of compass and a pencil to draw any circle in your exercise books.
2. Draw a straight line that passes through the centre of the circle and measure the distance from the centre of both ends.
3. Cut off the circle you have drawn using a razor blade or scissors.
4. Label its parts.

The basic elements of a circle are:

(a) Centre – Is a point inside the circle and is at an equal distance from the point on the circumference.

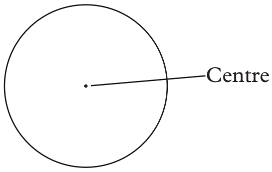


Fig 9.2

(b) Diameter – Is the distance across the circle through the centre.

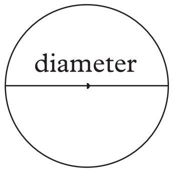


Fig 9.3

(c) Radius – Is the distance from the centre of the circle to any point on the circumference.

Radius is half the diameter.

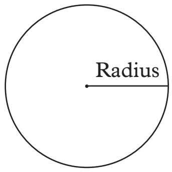
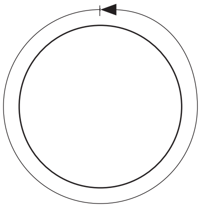


Fig 9.4

$$r = \frac{1}{2}d \Rightarrow d = 2r$$

(d) Circumference – This is the distance around the circle.



Circumference

Fig 9.5

(e) Chord – Is a straight line joining any two points on the circumference points of a circle. Diameter is a chord passing through the centre.

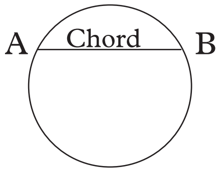


Fig 9.6

(f) Segment– is a region of circle that is cut of from the rest of the circle by a chord or secant. A chord or secant divides the circle into two segments.

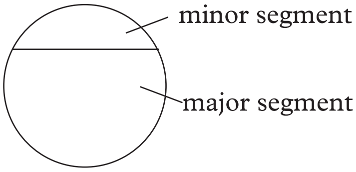


Fig 9.7

(g) Area of circle – Is the region enclosed by the circle.

(h) Tangent – Is a line touching the circumference of a circle only at one point and does not cut the circumference. The point at which it touches the circumference is called the **point of contact**.

A tangent can be drawn at any point on the circumference. Thus a circle can have an infinite number of tangent lines.

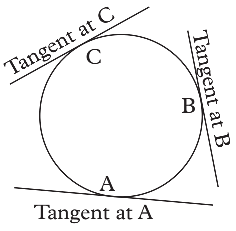


Fig 9.8

(i) Secant – Is a line that intersects a circle at two points.

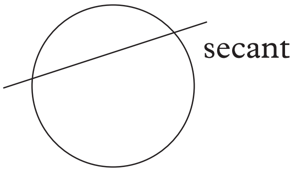


Fig 9.9

(j) Sector – Is the part of the circle that is enclosed by two radii of a circle and their interrupted arc.

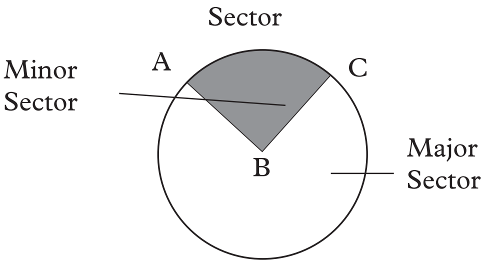


Fig 9.10

(k) Arc – Is a part of circumference. The shorter one AB is minor arc while longer arc between AB is major arc.

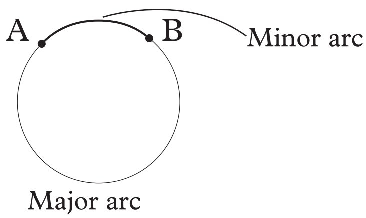


Fig 9.11

9.2 Circle theorem

9.2.1 Angles at the centre and the circumference of a circle

Activity 9.2

1. Draw a circle of centre O with any convenient radius.
2. Mark two points A and B on the circumference such that AB is a minor arc.
3. Mark another point P on the major arc AB.
4. Draw angles AOB and APB.
5. Measure ∠AOB and ∠APB.
6. What is the relationship between the two angles?
7. Compare your observations with those of other members of your class. What do you notice?

If AB is an arc and O is the centre of a circle (Fig. 9.12), angle AOB is called the central angle of the circle, subtended by minor arc AB.

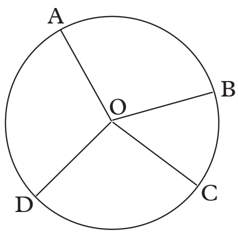


Fig. 9.12

Similarly, angles BOC, COD and AOD are examples of central angles of the circle.

If two central angles in the same circle, or in equal circles, are equal, then the arcs that subtend them must be equal.

Consider a case in which a chord or arc subtends a central angle (at the centre of circle) and another angle at circumference of the circle in the same segment. Fig. 9.13.

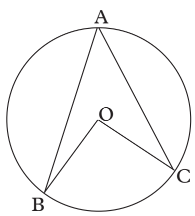


Fig. 9.13

The point O in the centre of the circle and A, B and C lie on the circumference. By measurement, ∠BOC = 2∠BAC

Similary in Fig 9.14 shown below.

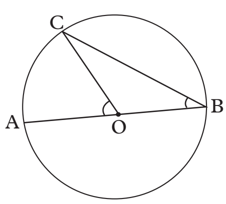


Fig. 9.14

∠AOC = 2∠ABC

Theorem 1

Angle subtended at the centre of a circle is twice the angle subtended by the same chord or arc on the circumference of the same circle in the same segment.

Proof

Given: Circle ABC, centre O

∠BOC at the centre

∠BAC on the circumference standing on the same arc BC.

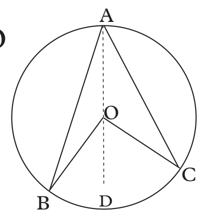


Fig. 9.15

To prove: ∠BOC = 2∠BAC

Join AO and produce it to a point D.

In ∠OAB, OB = OA .... radii

∠OAB = ∠OBA .... base angles of an isosceles triangle

Exterior ∠BOD = ∠OBA + ∠OAB

∴ ∠BOD = 2∠OAB

Similarly in ∠OAC, OA = OC .... radii

∴ ∠OAC = ∠OCA

Exterior ∠COD = ∠OAC + ∠OCA

∠COD = 2 ∠OAC

∠BOC = ∠BOD + ∠COD

= 2∠OAB + 2∠OAC

= *2 (OAB + OAC)*

∴ *∠BOC = 2 ∠(BAC)*

Note:

This theorem also relates the reflex angle subtended at the centre by a chord with the angle subtended by the same chord in the minor segment as shown in Fig 9.16 below.

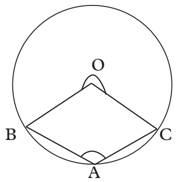


Fig. 9.16

∠BOC = 2∠BAC

Example 9.1

PQ is the diameter of a circle centre O (Fig. 9.16). R is a point on the circumference such that ∠POR = 48°. Find ∠PQR.

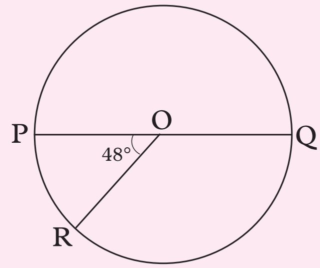


Fig. 9.17

Solution

Join R to Q (Fig. 9.17)

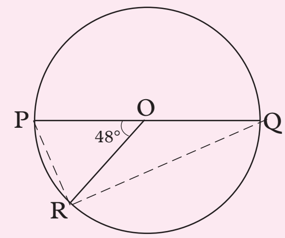


Fig. 9.18

$\angle PQR = \frac{1}{2}\angle POR = 24°$ (Angle at the centre is twice angle at the circumference)

Example 9.2

Figures 9.19 and 9.20, shows, circles with their centres marked O. AB, BC and *AC are chords of the circles in each case. Determine the size of the angles marked x, y and z.*

(a)

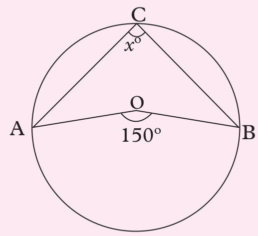


Fig. 9.19

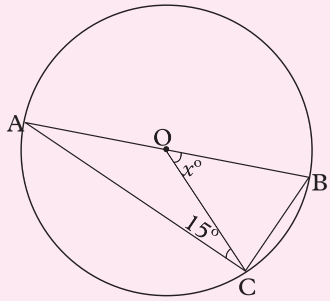


Fig. 9.20

Solution

(a) *In Fig. 9.19 ∠AOB is subtended at the centre while ∠ACB is subtended at the circumference, both by arc AB (or chord AB).*

$\angle ACB = \angle x = \frac{AOB}{2} = \frac{150°}{2} = 75°$

(b) *In Fig. 9.20 ∠COB and ∠CAB are subtended by chord CB, at the centre and circumference respectively.*

OAC is isosceles hence ∠ACO

$= \angle CAB = 15°$

$\therefore \angle COB = x = \angle OAC \times 2$

$= 15° \times 2 = 30°$

Example 9.3

In Fig. 9.21, ∠ABD = 58° and CBD is a straight line. Calculate:

(a) *obtuse ∠AOC,* (b) *∠AEC.*

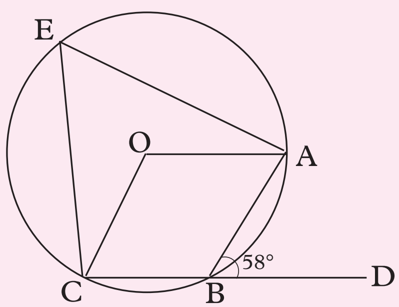


Fig. 9.21

Solution

(a) *∠ABC = 180° – 58° = 122° (Angles on a straight line.)*

Now reflex ∠AOC = 122° × 2 = 244° (Angle subtended at the centre by major arc AC.).

Thus,

Obtuse ∠AOC = 360° – 244°

= 116° (Angles at a point.)

(b) *∠AEC is subtended at the circumference by the minor arc ABC.*

$\therefore 2\angle AEC = \text{obtuse } \angle AOC$

$= 116°$

$\therefore \angle AEC = \frac{116°}{2}$

$= 58°$

Exercise 9.1

1. In Fig. 9.22 below, O is the centre and AC is the diameter.

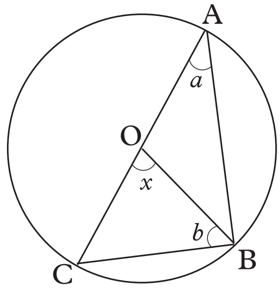


Fig. 9.22

(a) If $a = 24°$, find x.

(b) If $x = 62°$, find b.

(c) If $a = 52°$, find b.

(d) If $b = 28°$, find a.

(e) Express x in terms of a.

(f) Express b in terms of a.

2. In Fig. 9.23, O is the centre of the circle. If $c = 47°$, find d and p.

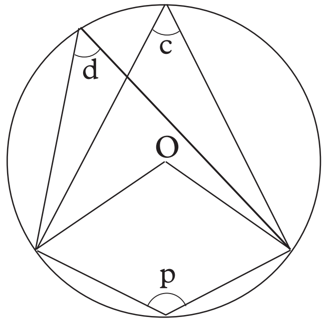


Fig. 9.23

3. AB is the diameter of a circle and C is any point on the circumference. If ∠BAC = 2∠CBA, find the size of angle BAC.

4. ABC is a triangle such that points A, B and C lie on circumference of a circle, centre O. If AC = BC and angle AOB = 72°, find angle BAC.

9.2.2 Angle in a semicircle

Activity 9.3

1. Draw a circle, centre O, using any convenient radius.
2. On your circle draw a diameter AB.
3. Mark another point C on the circumference and join A to C and B to C.
4. Measure angle ACB.
5. Compare your result with those of other members of your class. What do you notice?

We can apply what we have just learned about the angle at the centre of a circle and prove that the angle in a semicircle is always a right angle.

Theorem 2

The angle subtended by the diameter at any point on the circumference of a circle is a right angle.

Proof

Given: Circle ADB centre O, diameter AB.
C is any point on the semicircle ACB

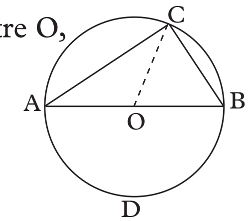


Fig. 9.24

To prove: ∠ACB = 90°
i.e. it is a right angle.

Join OC
OA = OC …… radii
∠OCA = ∠OAC … base angles of an isosceles Δ
Also OB = OC

∴ ∠OBC = ∠OCB …base angles of an isosceles Δ

∴ ∠ACB = ∠OAC + ∠OBC
But ∠ACB + ∠OAC + ∠OBC = 180° = 2 right angles.

∴ ∠ACB = (∠OAC + ∠OBC)

$= \frac{1}{2}$ of 180°

$= \frac{1}{2}$ of 2 right angles

= 90°

= 1 right angle.

Also, from theorem 1, angle at the centre is twice angle at the circumference,

Angle O = 180° = $\frac{1}{2}$ angle ACB
= 90°

Example 9.4

In the figure below, calculate giving reasons, the angles marked with letters if O is the centre of the circle.

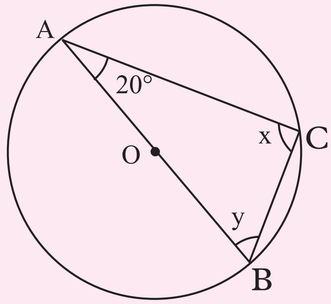


Fig. 9.25

Solution

$\angle x = 90°$, *angle subtended by the diameter on the circumference.*

$\angle y = 180° – (x + 20)$, *sum of angles in a triangle add up to 180°.*

$\angle y = 180° – 110° = 70°$

Example 9.5

Fig 9.26 shows a circle centre O, diameter BOY. Triangle ABC is isosceles and $\angle ACB = 63^o$. Use the given information to find angles:

(a) *BAC* (b) *ACY,*
(c) *CBY,* (d) *ABY*
(e) *BYC*

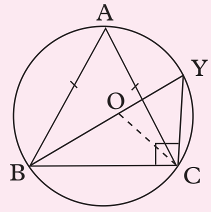


Fig. 9.26

Solution

(a) *Using* ΔABC, $\angle ABC = \angle ACB = 63°$

$\therefore \angle BAC = 180 – (\angle ABC + \angle ACB)$

$= 180° – 126° = 54°$

$\therefore \angle BAC = 54°$

(b) *Using* ΔBCY,

$\angle BCY = 90°$ …. *angle on a semicircle*

$\angle ACY = \angle BCY – \angle ACB$

$= 90° – 63° = 27°$

$\therefore \angle ACY = 27°$

(c) *Join O to C so that BOC is an isosceles triangle.*

Thus: $\angle BOC = 2\angle BAC = 2 \times 54°$

$= 108°$

$\therefore \angle CBO = \angle BCO$

$= \frac{180° – 108°}{2} = \frac{72^o}{2}$

$= 36°$

$\angle CBO = \angle CBY = 36°$

(d) $\angle ADY = 63° – 36° = 27°$

(e) $\angle BYC = 180° – (90° + 36°)$

$= 180° – 126°$

$= 54°$

Exercise 9.2

1. Calculate the angle marked *t* in Figure 9.27.

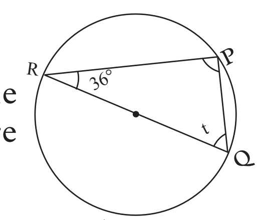


Fig. 9.27

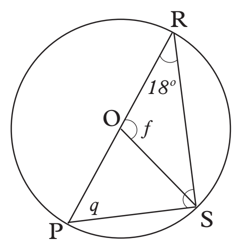


Fig. 9.28

2. Figure 9.28 is a circle centre O. If POR is the diameter, calculate the value of:
   (a) Angle marked q.
   (b) angle marked f.

3. A circle centre O is divided by chord AOC. BX, CD and AD are other chords. If ∠BDC = 65°, calculate the value of:
   (a) angle a
   (b) angle b
   (c) angle c

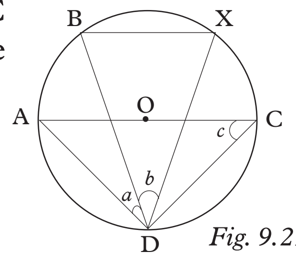


Fig. 9.29

9.2.3 Angles in the same segment

Activity 9.4

1. Draw a circle centre O, with any convenient radius.
2. Mark off a minor arc AB.
3. On the major arc AB, mark distinct points P, Q, R and S.
4. Join each of the points in (3) above so as to form angles APB, AQB, ARB and ASB.
5. Measure the angles.
6. What do you notice about the sizes of the four angles?

   Do your classmates have the same observation?

We have already learnt that, a chord divides the circumference into two arcs, a major and a minor arc.

If AB is a minor arc and P is any point on the major arc (Fig. 9.30), then ∠APB is the angle subtended by the minor arc AB at the circumference of the circle.

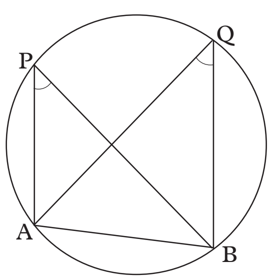


Fig. 9.30

If Q is another point on the major arc AB (Fig. 9.30), ∠APB and ∠AQB are both subtended at the circumference by the minor arc AB or chord AB. Such angles are said to be in the same segment.

Consider Fig. 9.31 below which ∠AEB and ∠AGB are subtended by chord AB onto the circumference on the major segement.

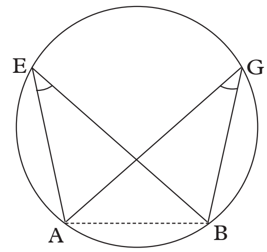


Fig. 9.31

By measurement, ∠AEB = ∠AGB

Similarly, consider Fig. 9.32 below ∠ACB and ∠ADB that are subtended by chord AB or by the major arc AB at the circumference. These are also angles in the same segment. Are they also equal? Check by measuring.

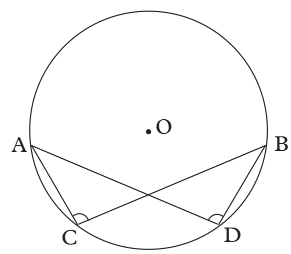


Fig. 9.32

∠ACD = ∠ADB

Theorem 3

Angles subtended on the circumference by the same chord in the same segment are equal.

Proof

Given: Circle centre O

Chord BC (Fig. 9.33)

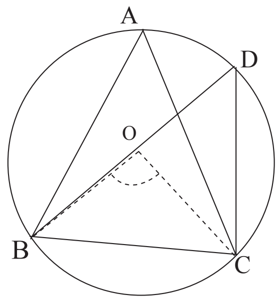


Fig. 9.33

Let ∠BAC and ∠BDC be the angles subtended by the chord BC on the major arc BADC.

To prove: that ∠BAC = ∠BDC

Construction: Join BO, OC.

Proof: ∠BOC = 2 ∠BAC ....*angle at the centre of a circle is twice the angle subtended by the same chord on the circumference.*

Similarly, ∠ BOC = 2 ∠BDC.

∴ ∠BAC = ∠BDC.

Activity 9.5

1. Draw a circle with any convenient radius.
2. On the circumference, mark points A, B, C and D such that chord AB has the same length as chord CD.
3. On major arc AB mark point P and on major arc CD mark point Q.
4. Draw angles APB and CQD.
5. Measure the angles APB and CQD.
6. What do you notice about the sizes of the angles?
7. Using a piece of sewing thread, measure the lengths of the minor arcs AB and CD. What do you notice about their lengths?

In this activity, we also learn that:

1. Equal arcs of the same circle subtend equal angles at the circumference.
2. Equal chords of the same circle cut off equal arcs.

Example 9.6

In Fig. 9.34 below, find the size of angle y.

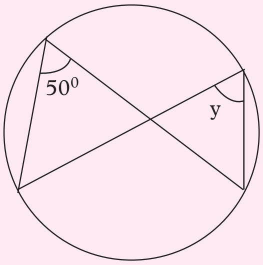


Fig. 9.34

Solution

Angle y = 50⁰
Angle obtained by the same arc at the circumference are equal.

Example 9.7

Fig. 9.35 shows a circle centre O. Find the values of the angles marked with letters.

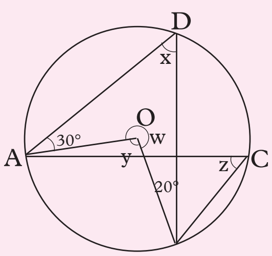


Fig. 9.35

Solution

Join D to the centre O of the circle so that DO produced meets the circumference at a point P (Fig. 9.36).

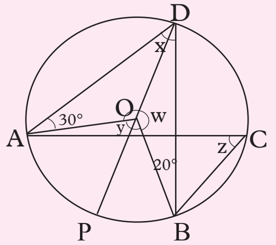


Fig. 9.36

Since OA, OD and OB are radii of the same circle, OA = OD = OB.

Thus, Δs AOD and BOD are isosceles.

$\therefore$ *In ΔAOD,* $\angle DAO = \angle ADO = 30°$
(Base angles of an isosceles Δ)

In ΔBOD, $\angle DBO = \angle BDO = 20°$
(Base angles of an isosceles Δ)

$\therefore \quad \angle ADB = x = 30° + 20°$
$= 50°$

In ΔADO, $\angle DAO + \angle ADO = \angle AOP$
(Exterior angle of a Δ is equal to the sum of the opposite interior angles).

$\therefore \quad 30° + 30° = \angle AOP$
i.e. $\angle AOP = 60°$

Similarly, $\angle BOP = \angle DBO + \angle BDO$
$= 20° + 20°$
$= 40°$

But $\angle AOP + \angle BOP = y$
$\therefore \quad 60° + 40° = y$
i.e. $y = 100°$
$w + y = 360°$ *(Angles at a point add up to 360°)*
$\therefore \quad w = 360° - 100°$
$= 260°$
$\angle ADB = \angle ACB$ *(Angles subtended by the same minor arc AB in the same segment.)*
$\angle ACB = 50°$
i.e. $z = 50°$

Example 9.8

ABCDE is a regular pentagon inscribed in a circle (Fig. 9.37). Show that $\angle ACD = 2\angle ACB$.

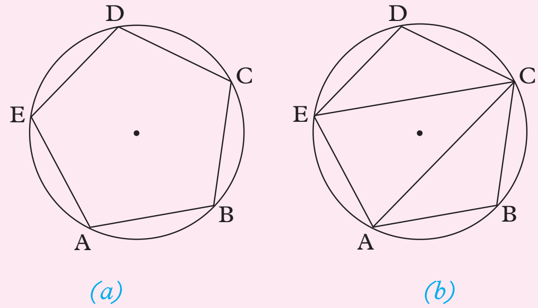


Fig. 9.37

Solution

With reference to Fig. 9.37 (b), $\angle ACB$ *is subtended at the circumference of the circle by the minor arc AB.*

$\angle ECA$ *is subtended at the circumference of the circle by the minor arc AE.*

$\angle DCE$ *is subtended at the circumference of the circle by the minor arc DE.*

But arcs AB, AE and DE are equal. Therefore, they subtend equal angles at the circumference.

$\therefore \quad \angle ACB = \angle ECA = \angle DCE$

But $\angle ACD = \angle ACE + \angle ECD$

$\therefore \quad \angle ACD = 2\angle ACB$

Exercise 9.3

1. Find x and y in Fig. 9.38 below.

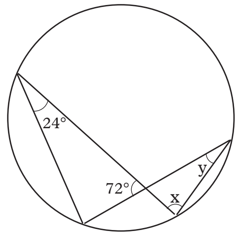


Fig. 9.38

2. What are the values of a and b in Fig. 9.39 below?

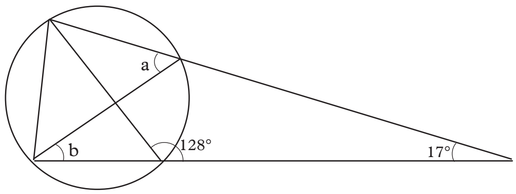


Fig. 9.39

3. (a) Determine the angles in ΔABC in Fig. 9.40.

   (b) What does your result tell you about the minor arcs AC and BC?

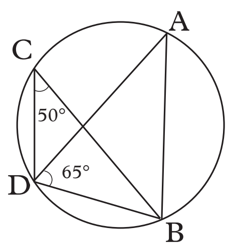


Fig. 9.40

4. Determine the value of angles x and y in the Fig. 9.41 below if O is the centre of the circle.

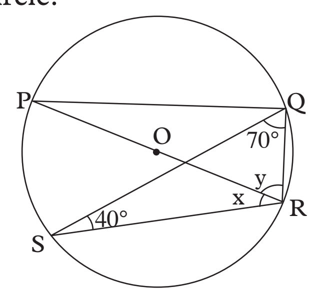


Fig. 9.41

5. In the Fig. 9.42 below, O is the centre of the circle. Find the values of angles x and y.

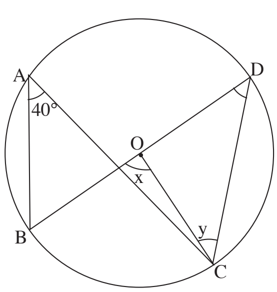


Fig. 9.42

6. Find the values of angles x and y in the Fig. 9.43 below.

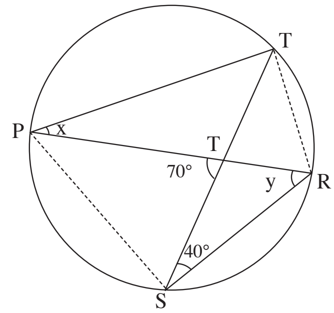


Fig. 9.43

9.2.4 Angles in a cyclic quadrilateral

Activity 9.6

1. Draw a circle centre O using any convenient radius.
2. On the circumference, mark points A, B, C and D in that order and join them to form a quadrilateral.
3. Measure angles ABC and ADC. Find their sum.
4. Measure angles BAD and BCD. Find their sum.
5. What do you notice about the two sums in 3 and 4?
6. Are the pairs of angles in 3 and 4 adjacent or opposite?
7. Do the other members of your class have the same observations as you do?
8. Produce side AB of the quadrilateral, and measure the exterior angle so formed. How does the size of this angle compare with that of interior ∠ADC?

Consider Fig 9.44 showing a cyclic quadrilateral ABCD.

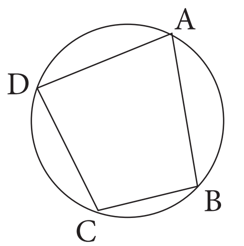


Fig. 9.44

Angles BAD, CBA, DCB and ADC are the interior angles of the quadrilateral.

In Fig 9.45 shows exterior angles *a*, *b*, *c* and *d* of the quadrilateral ABCD.

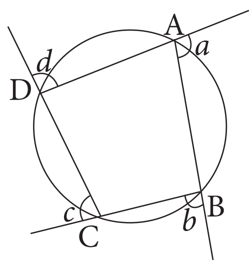


Fig. 9.45

A cyclic quadrilateral is a quadrilateral whose vertices all lie on a circle. The distinctive property of a cyclic quadrilateral to be looked under this theorem is that its opposite angles are suplementary.

Theorem 4.1

The opposite interior angle of a cyclic quadrilateral are supplementary or add upto 180°.

Proof

Given: Quadrilateral ABCD inscribed in a circle centre O.

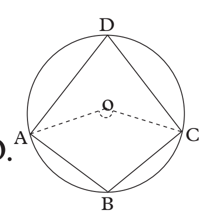


Fig. 9.46

To prove:

(i) $\angle ADC + \angle ABC = 2$ right angles

(ii) $\angle BAD + \angle BCD = 2$ right angles

Join OA and OC.

$\angle ADC = \frac{1}{2}\angle AOC$ ....$\angle ADC$ and $\angle AOC$ subtended by the arc ABC at the circumference and centre of the circle respectively.

Similarly,

$\angle ABC = \frac{1}{2}$ reflex $\angle AOC$

$\angle ABC + \angle ADC = \frac{1}{2}\angle AOC + \frac{1}{2}$ reflex $\angle AOC$

$= \frac{1}{2}$ (AOC + reflex $\angle AOC$)

$= \frac{1}{2}(360^\circ)$

$= \frac{1}{2}$ four right angles

$\therefore$ $\angle ADC + \angle ABC = 2$ right angles

Similarly, $\angle BAD + \angle BCD = 2$ right angles

$\therefore$ Opposite angles of a cyclic quadrilateral are supplementary.

Theorem 4.2

If one side of a cyclic quadrilateral is produced, the exterior angle formed is equal to the opposite interior angle of the quadrilateral.

Proof

Given: Quadrilateral ABCD inscribed in a circle centre O.

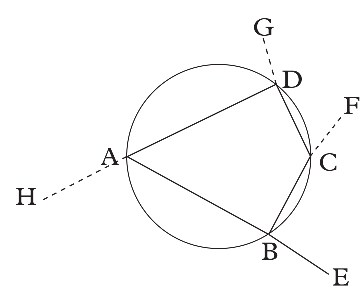


Fig 9.47

To prove:

(i) ∠ADC = the exterior angle at B.

(ii) ∠BAD = the exterior angle at C

(iii) ∠DCB = exterior angle at A

(iv) ∠ABC = exterior angle at D

Produce line AB to a point E.

∠ABC + ∠CBE = 2 right angles

………. angles on a straight line

∠ABC + ∠ADC = 2 right angles ……..

opposite angles of a cyclic quadrilateral

∴ ∠CBE = ∠ADC

Similarly, we can produce lines BC, CD and DA to points F, G, H respectively, and prove that

∠BAD = ∠DCF

∠ABC = ∠ADG

Note: A cyclic quadrilateral can be drawn without the circle so long as condition 1, 2 or 3 is satisfied.

Example 9.9

Find the angles x and y in the figure below.

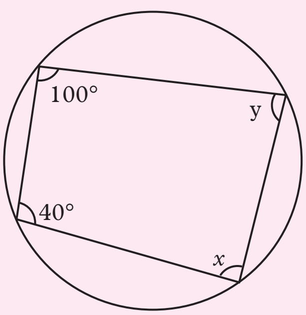


Fig 9.48

Solution

$\angle x = 180° - 100°$ (Opposite angles in a cyclic quadrilateral add up to 180°.)

$x = 180°$

$y = 180° - 40°$ (Opposite angles in a cyclic quadrilateral add up to 180°.)

$y = 140°$

Example 9.10

Find angles a and b in the Fig 9.49 below.

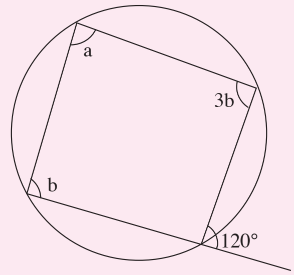


Fig. 9.49

Solution

a = 120° (Exterior angle of a cycliclic quadrilateral is equal to opposite interior angle.)

b + 3b = 180°, opp. angles of a cyclic quadrilateral.

$4b = 180°$

$b = 45°$

Example 9.11

In quadrilateral ABCD (Fig.9.50), ∠ADC = 122°, ∠BDC = 41° and ∠ACB = 81°. Show that the quadrilateral is cyclic.

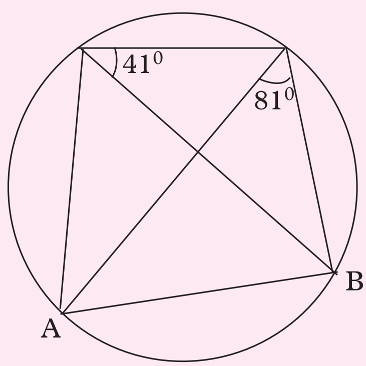


Fig. 9.50

Solution

$$\angle ADB = \angle ADC - \angle BDC$$
$$= 122° - 41°$$
$$= 81°$$
$$\therefore \quad \angle ADB = \angle ACB.$$

But these angles are subtended by the same side AB at the points C and D, on the same side of AB.

Therefore ABCD is a cyclic quadrilateral and so, point A, B, C and D are concyclic.

Example 9.12

In Fig. 9.51, find the values of x and y.

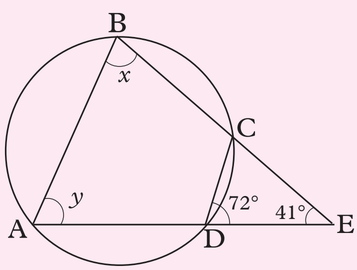


Fig. 9.51

Solution

$\angle ADC = 180° - 72° = 108°$ *(Angles on a straight line).*

$x + 108° = 180°$ *(Opposite angles of a cyclic quadrilateral)*

$\therefore \; x = 180° - 108° = 72°.$

Note that $\angle ABC$ *is the interior angle opposite to the exterior angle at D.*

$\angle DCE = 180° - (72° + 41°)$ *(angle sum of a triangle)*

$= 180° - 113°$

$= 67°$

$\therefore \; y = \angle DCE = 67°$ *(exterior angle equals interior opposite angle).*

Exercise 9.4

1. A, B, C, D and E are five points, in that order, on the circumference of a circle (Fig. 9.52).

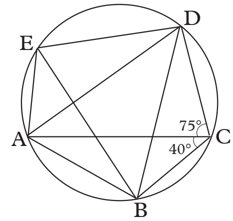


Fig. 9.52

   (a) Write down all angles in the figure equal to $\angle ACB$.
   (b) Write down all angles in the figure supplementary to $\angle BCD$.
   (c) If $\angle ACB = 40°$ and $\angle ACD = 75°$, find the size of $\angle DEB$.

2. In Fig. 9.53, find:
   (a) $\angle BCD$ (b) $\angle CDA$.

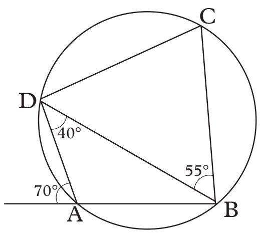


Fig. 9.53

3. Fig. 9.54 consists of two intersecting circles. Use it to find the angles marked by letters.

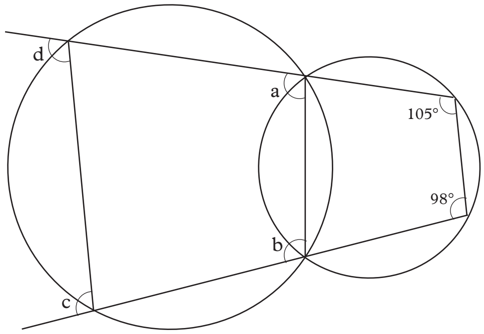


Fig. 9.54

4. ABCD is a cyclic quadrilateral in a circle centre O.
   (a) If ∠ABD = 32°, find ∠ACD.
   (b) If AOC is a straight line, write down the size of ∠ABC.
5. AB is a chord of a circle centre O. If ∠AOB = 144°, calculate the angle subtended by AB at a point on the minor arc AB.

9.2.5 Tangent to a circle

9.2.5.1 Definition of tangent to a circle

Activity 9.7

1. Draw a circle of radius 10 cm.
2. Using a ruler and pencil, carefully draw lines that touch the circumference at only one point. What is the name of such a line.

Consider Fig. 9.55

In Fig. 9.55(a), lines KL and PQ have only one point common with the circle. A line with at least one point common with the circle is said to **meet** the circle at that point.

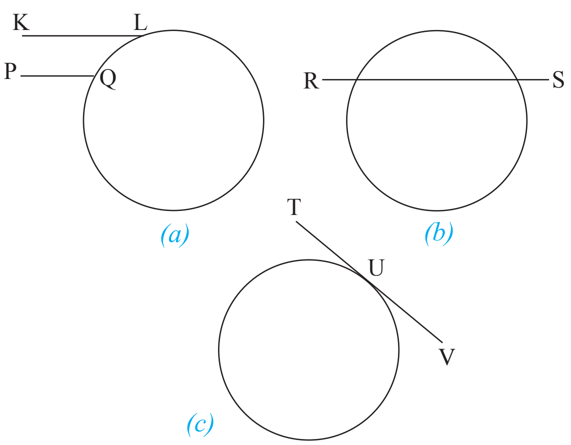


Fig. 9.55

In Fig. 9.55(b), line RS has two distinct points common with the circle. Such a line is said to **meet and cut** the circle at the two points.

In Fig. 9.55(c), the line TV has one point of contact with the circle. Line TV is said to **meet and touch** the circle at that point of contact. Point U is called the **point of contact**.

Note:

1. A line which cuts a circle at two distinct points (as in Fig. 9.45(b)) is called a **secant** of the circle.
2. A line which has one, and only one point in contact with a circle (as in Fig. 9.55(c)), however far it is produced either way, is called a **tangent** of the circle.

Activity 9.8

1. Draw a circle of radius 10 cm on a manilla paper.
2. As accurately as possible, draw a tangent to the circle at a point of your choice.
3. Join point P to the centre ofthe circle.
4. Measure the angle between the tangent and radius.
5. Repeat this for other tangents drawn at other points on the circumference.

What do you conclude.

Consider Fig. 9.56.

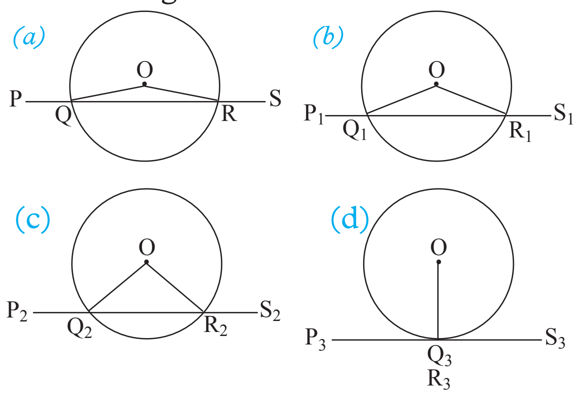


Fig. 9.56

Fig. 9.56(a) to (d) shows what happens when the secant PQRS moves away from the centre of the circle. As the secant moves further away, the points Q and R get closer to each other and the chord QR gets shorter each time. Eventually Q and R coincide at one point (Fig. 9.56(d)). On the other hand, angles OQP and ORS become smaller and smaller. Eventually when Q and R coincide, angles OQR and ORS each becomes 90°.

Note that in ΔOQR, since OQ = OR,
∠OQR = ∠ORQ.

It follows that ∠PQO = ∠SRO.

Therefore, when Q and R coincide (Fig. 9.56(d)), ∠PQO =∠SRO = 90°.

Hence the radius is perpendicular to the tangent PS.

Theorem 5

1. A tangent to a circle is perpendicular to the radius drawn through the point of contact.
2. Conversely stated the perpendicular to a tangent at its point of contact passes through the centre of the circle.

Example 9.13

In Fig. 9.57, AC is a tangent to the circle, centre O. If ∠ABD = 120°,

(a) what is the size of ∠ODB?

(b) what is the length of OA if OB = 6 cm and AB = 7.5 cm?

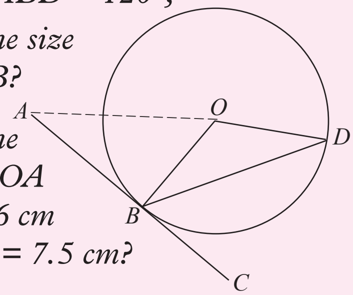


Fig. 9.57

Solution

(a) Since AC is a tangent and OB is a radius, AC is perpendicular to OB.

$\therefore \angle ABO = 90°.$

$\angle OBD = \angle ABD - \angle ABO$

$= 120° - 90°$

$= 30°$

Since $OB = OD$ *(radii)*

$\angle ODB = \angle OBD$ *(base angles of an isosceles Δ)*

$\therefore \angle ODB = 30°$

(b) $OA^2 = AB^2 + BO^2$ *(right angle Δ; Pythagoras theorem)*

$= 7.5^2 + 6^2$

$= 92.25$

$\therefore OA = 9.605 \text{ cm} \approx 9.6 \text{ cm (1 d.p.)}$

Example 9.14

In Fig. 9.58, C, D and E are points on a circle centre O. AB is a tangent to the circle at E, AOC is a straight line and ∠CAE = 20°. Find the size of ∠CDE.

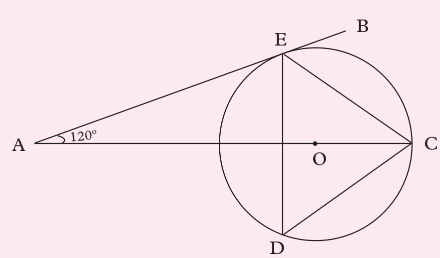


Fig. 9.58

Solution

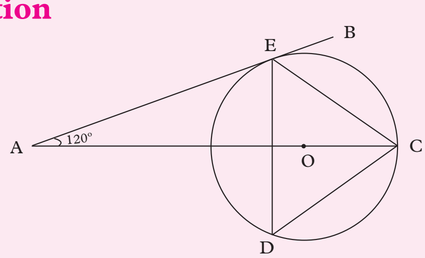


Fig. 9.59

Draw line OE.

∠AEO = 90° (Angle between a tangent and a radius)

∠AOE = 180° – (20° + 90°) (Angle sum of a triangle.)

= 70°

∠COE = 180° – 70° (Angles on a straight line.)

= 110°

$\therefore \angle CDE = \frac{1}{2} \times \angle COE = \frac{1}{2} \times 110^o = 55^\circ$

(Angle subtended on cirmumference equals $\frac{1}{2}$ *× angle subtended at centre).*

Exercise 9.5

1. In the Fig. 9.60 below, O is the centre of the circle while A and C are points on the circumference of the circle. BCO is a straight line and BA is a tangent to the circle. AB = 8cm and OA = 6cm.

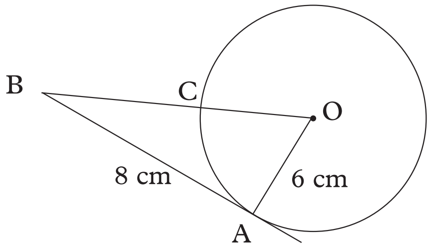


Fig. 9.60

   (a) Explain why ∠OAB is a right angle.
   (b) Find the length BC.

2. Fig. 9.61 shows a circle,centre O. PR is a tangent to the circle at P and PQ is a chord.

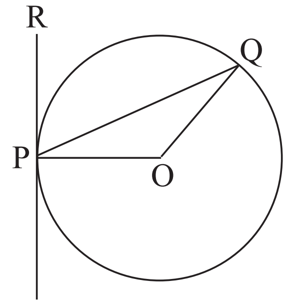


Fig. 9.61

   Calculate:
   (a) ∠RPQ given that ∠POQ = 85°.
   (b) ∠RPQ given that ∠PQO = 26°.
   (c) ∠POQ given that ∠RPQ = 54°.
   (d) ∠POQ given that ∠QPO = 17°.

3. In Fig. 9.62, ABC is a tangent and BE is a diameter to the circle.

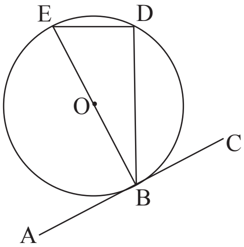


Fig. 9.62

   Calculate:
   (a) ∠EBD if ∠CBD = 33°.
   (b) ∠BED if ∠ABD =150°.
   (c) ∠DBC if ∠DEB = 65°.
   (d) ∠ABD if ∠BED = 38°.

4. Two circles have the same centre O, but different radii. PQ is a chord of the bigger circle but touches the smaller circle at A. Show that PA = AQ.

5. Two circles have the same centre O and radii of 13 cm and 10 cm. AB is a chord of the bigger circle, but a tangent to the small circle. What is the length of AB?

6. A tangent is drawn from a point 17 cm away from the centre of a circle of radius 8 cm. What is the length of the tangent?

9.2.5.2 Constructing a tangent at any given point on the circle

Activity 9.9

1. Draw a circle, centre O, using any radius.
2. Draw a line OB through any point A on the circumference, with B outside the circle.
3. At A, construct a line PQ perpendicular to OB.

To construct a tangent to a circle, we use the fact that a tangent is perpendicular to the circle at the point of contact.

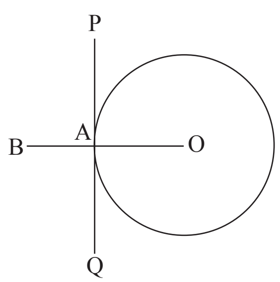


Fig. 9.63

The line PQ (Fig. 9.63) is a tangent to the circle at A.

9.2.5.3 Constructing tangents to a circle from a common point

Activity 9.10

1. Draw a circle of any radius and centre O.
2. Mark a point T outside the circle.
3. Join OT. Construct the perpendicular bisector of TO to meet TO at P.
4. With centre P, radius PO, construct arcs to cut the circle at A and B.
5. Join A to T and B to T. What do you notice?

Activity 9.11

1. Draw a circle of any radius, centre O.
2. Choose any points A and B on the circle. Construct tangents at A and B
3. Produce the tangents till they meet at a point T.
4. Join OA, OB and OT.
5. Measure: (a) AT, BT.
   (b) ∠ATO, ∠BTO,
   (c) ∠AOT, ∠BOT
6. What do you notice?
7. Which points on a circle would have tangents that do not meet?

If two tangents are drawn to a circle from a common point:

(a) the tangents are equal;
(b) the tangents subtend equal angles at the centre;
(c) the line joining the centre to the common point bisects the angles between the tangents.

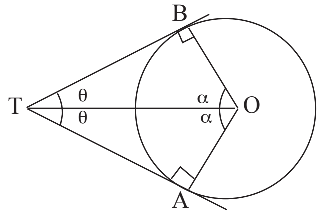


Fig. 9.64

Example 9.15

In the figure below, triangle ABC is the tangent to the circle of centre O at two points N and B.

Length AM = 6 cm, length BC=18 cm and the radius of the circle is 9 cm

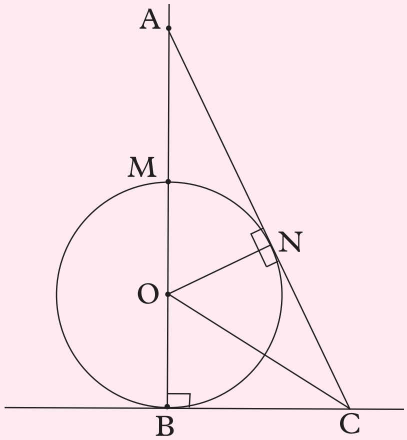


Fig. 9.65

Find the following;

(a) length AN
(b) length AC
(c) length OC

Solution

(a) *From triangle ONA, we see that it is right angled at N*

Since ON = 9 cm (radius) = OM

$OA = OM + AM$
$OA = 9\ cm + AM$
$OA = 15\ cm$

By pythagoras theorem,

$ON^2 + AN^2 = OA^2$

$9^2 + AN^2 = 15^2$

$AN^2 = 225 - 81$

$AN^2 = 144$

$AN = \sqrt{144}$

$AN = 12\ cm$

(b) *For triangle ONC, On = 9 cm,*

$BC = NC = 18\ cm$

$\therefore AC = AN + NC$

$AC = 12\ cm + 18\ cm$

$AC = 30\ cm$

Or

$AC^2 = AB^2 + BC^2$

$AC^2 = 24^2 + 18^2$

$AC^2 = 900$

$AN = \sqrt{900}$

$AN = 30\ cm$

(c) $OC^2 = OB^2 + BC^2$

$OC^2 = 9^2 + 18^2$

$OC^2 = 81 + 324$

$OC^2 = 405$

$OC = \sqrt{405} = 20.12\ cm$

Example 9.16

In Fig. 9.66, TA and TB are tangents to the circle, centre O.

If $\angle ABO = 28°$, what is the size of $\angle ATO$?

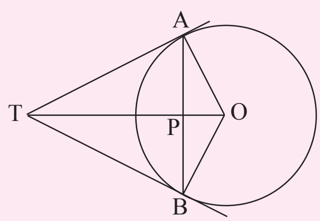


Fig. 9.66

Solution

In ΔABO,

$\angle ABO = \angle BAO = 28°$ (isosceles Δ)

$\therefore \angle AOB = 180° - 56° = 124°$

$\angle AOT = \frac{1}{2}\angle AOB$ *(Tangents subtend equal angles at the centre of circle.)*

$= 62°$

In ΔATO, $\angle OAT = 90°$ *(Tangent is perpendicular to radius.)*

$\therefore \angle ATO = 90° - \angle AOT$

$= 90° - 62°$

$= 28°.$

Exercise 9.6

1. In Fig 9.67 below, A, B and D are points on the circumference of a circle centre O. BOD is the diameter of the circle while BC and AC are the tangents of the circle. Angle OCB = 34°. Work out the size of angle DOA.

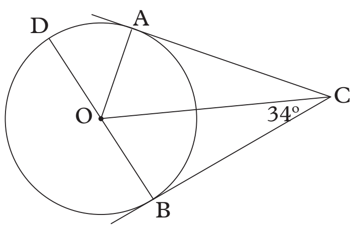


Fig 9.67

2. In Fig 9.68 A and B are points on the circumference of a circle, centre X. PA and PB are tangents to the circle. Angle APB = 86°. Find the size of angle *y*.

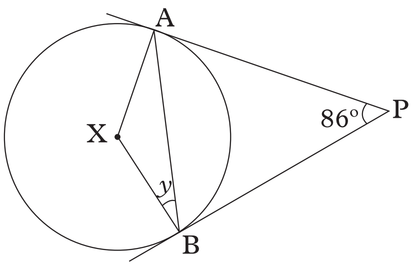


Fig. 9.68

3. In Fig. 9.69, O is the centre of the circle. PT and RT are tangents to the circle.

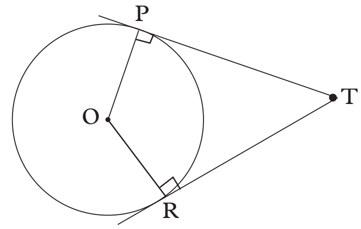


Fig. 9.69

Calculate:

(a) ∠POT if ∠OTR = 34°.
(b) ∠PRO if ∠PTR = 58°.
(c) ∠TPR if ∠PRO = 15°.
(d) ∠RTO if ∠POR = 148°.

4. Draw a circle, centre O, and radius 2.5 cm. Mark points A and B on the circle such that ∠AOB = 130°. Construct tangents at A and B. Measure:

(a) The lengths of the tangents.
(b) The angle formed where the tangents meet.

5. In Fig. 9.70, O is the centre of the circle. If BO = 19.5 cm, BQ = 18 cm, QC = 8.8 cm and AO = 9.9 cm, what are the lengths of:

(a) AB (b) BC (c) AC?

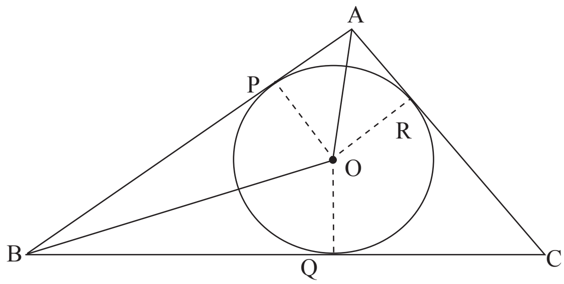


Fig. 9.70

6. A tangent is drawn to a circle of radius 5.8 cm from a point 14.6 cm from the centre of the circle. What is the length of the tangent?

7. Tangents are drawn from a point 10 cm away from the centre of a circle

of radius 4 cm. What is the length of the chord joining the two points of contact?

8. Tangents TA and TB each of length 8 cm, are drawn to a circle of radius 6 cm. What is the length of the minor arc AB?

9. Construct two tangents from a point A which is 6 cm from the centre of a circle of radius 4 cm.
   (a) What is the length of the tangent?
   (b) Measure the angle subtended at the centre of the circle.

10. Draw a line KL= 6 cm long. Construct a circle centre K radius 3.9 cm such that the tangent LM from L to the circle is 4.5 cm. Measure ∠KLM.

9.2.6 Angles in alternate segment

In Fig. 9.71, ABC is a tangent to the circle at B. Thechord BD divides the circle into two segments BED and BFD.

We say that BFD is the alternate segment to ∠ABD.

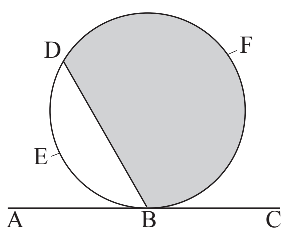


Fig. 9.71

Similarly, BED is the alternate segment to ∠CBD.

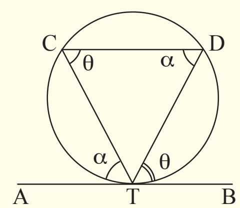
Activity 9.12

1. Draw a circle of any radius.
2. Draw a tangent at any point B.
3. Draw a chord BD.
4. Mark points P, Q, R on the circumference in the same segment. Join BP, BQ, BR, DP, DQ and DR.
5. Measure angles BPD, BQD and BRD.

   What do you notice?

Theorem 6

If a tangent to a circle is drawn, and from the point of contact a chord is drawn, the angle which the chord makes with the tangent is equal to the angle the chord subtends in the alternate segment of the circle. This is called the **alternate segment theorem**.



∠ATC = ∠CDT

∠BTD = ∠TCD

Fig. 9.72

Proof

(a) We use Fig. 9.73(a) to show that ∠RQT = ∠QST.

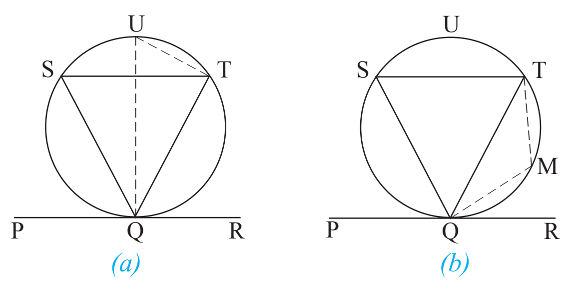


Fig. 9.73

Draw diameter QU. Join UT.

Since QU is a diameter and PR is a tangent,

$\angle RQT + \angle TQU = 90°$ (tangent ⊥ radius)

$\angle QTU = 90°$ (∠ in semi-circle)

$\therefore \angle QUT + \angle TQU = 90°$ (∠ sum of Δ)

$\therefore \angle RQT + \angle TQU = \angle QUT + \angle TQU$

$\Rightarrow \angle RQT = \angle QUT$.

But $\angle QUT = \angle QST$ (∠s in same segment)

$\therefore \angle RQT = \angle QST$.

(b) We use Fig. 9.69(b) to show that $\angle PQT = \angle QMT$

$\angle PQT + \angle RQT = 180°$ (Adj. ∠s on straight line.)

$\angle QMT + \angle QST = 180°$ (Opp. ∠s cyclic quadrilateral.)

$\therefore \angle PQT + \angle RQT = \angle QMT + \angle QST$.

But $\angle RQT = \angle QST$ (shown in Fig 9.69 (a) above)

$\therefore \angle PQT = \angle QMT$

Example 9.17

In Fig 9.74 below, ∠BAD = 40° and ∠BAC = 65°. What is y?

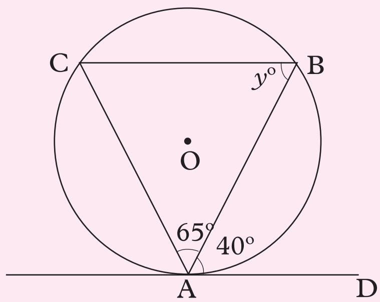


Fig 9.74

Solution

According to the tangent-chord theorem;

$< ACB = < BAD = 40°$

But $< ACB + < BAC + < ABC = 180°$ *(Angle sum of a triangle.)*

$\therefore < ABC = 180° - < ACB - < BAC$

$< ABC = 180° - < BAD - < BAC$

$< ABC = 180° - 40° - 65°$

$< ABC = 180° - 105°$

$< ABC = 75°$

$\therefore y = 75°$

Example 9.18

In Fig. 9.75, PQR is a tangent to the circle at Q. QT is a chord and PST is a straight line. Given that ΔPQT = 110°, ∠TPQ = 25°, find ∠SQP.

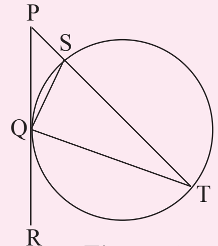


Fig. 9.75

Solution

In $\angle PQT$, $\angle PQT + \angle QTP + \angle TPQ = 180°$ *(∠sum of angles in a triangle)*

But $\angle PQT = 110°$, $\angle TPQ = 25°$

$\therefore \angle QTP = 180° - 135° = 45°$

But $\angle SQP = \angle QTP$ *(∠s in alternate segment)*

$\therefore \angle SQP = 45°$.

Example 9.19

In Fig. 9.76 line FEG is a tangent to the circle at point E. Line DE is parallel to CG, ∠DEC = 28° and ∠AGE = 32°.

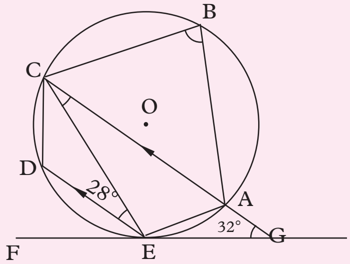


Fig. 9.76

Calculate:

(a) *∠AEG* *(b)* *∠ABC*

Solution

(a) *∠ACE = 28° (Alternate to ∠CED)*

∴ ∠AEG = 28° (Angles in alternate segments.)

(b) *∠DEF = 32° (Corresponding to ∠AGE.)*

∠DEG = 180° – 32° (Supplementary angle to ∠DEF.)

= 148°

∴ ∠CEA = 148° – (28° + 28°)

= 92°

Hence, ∠ABC = 180° – 92° (Opposite angles of a cyclic quadrilateral.)

= 88°

Exercise 9.7

1. If < BAD = 19°, find <ACB. in Fig 9.77.

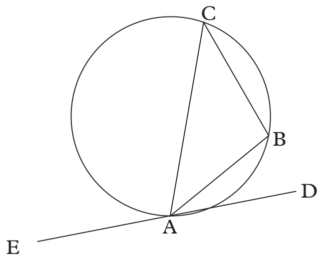


Fig. 9.77

2. In Fig. 9.78, AC is a tangent to the circle and BE//CD.

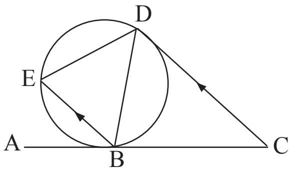


Fig. 9.78

   (a) If ∠ABE = 42°, ∠BDC = 59°, find ∠BED.
   (b) If ∠DBE = 62°, ∠BCD = 56°, find ∠BED.

3. In Fig. 9.79, PR is a tangent to the circle.

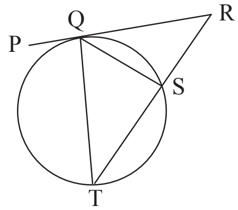


Fig. 9.79

   (a) If ∠PQT = 66°, find ∠QST.
   (b) If ∠QTS = 38° and ∠QRS = 30°, find ∠QST.
   (c) If ∠QTS = 35° and ∠TQS = 58°, find ∠QRS.
   (d) If ∠PQT = 50° and ∠PRS = 30°, find ∠SQT.

4. In Fig. 9.80, AB, BC and AC are tangents to the circle. If ∠BAC = 75° and ∠ABC = 44°, find ∠EDF, ∠DEF and ∠EFD.

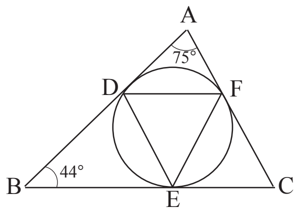


Fig. 9.80

5. In Fig. 9.81, KLM is a tangent to the circle. If ∠LPN = 38° and ∠KLP = 85°, find ∠PQN.

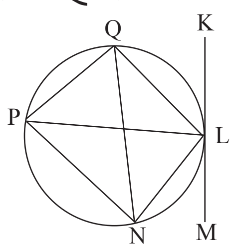


Fig. 9.81

6. In Fig. 9.82, DC is a tangent to the circle. Show that ∠CBD = ∠ADC.

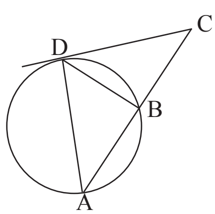


Fig. 9.82

7. In Fig. 9.83, AB and DE are tangents to the circle. ∠ABC = 40° and ∠BCD = 38°. Find ∠CDE.

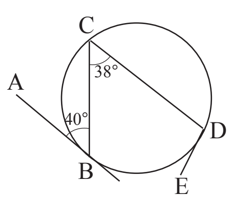


Fig. 9.83

8. In Fig. 9.84, ABC is a tangent to the circle at B and ADE is a straight line. If ∠BAD = ∠DBE, show that BE is a diameter.

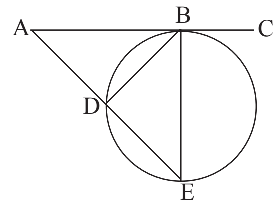


Fig. 9.84

9. In Fig. 9.85, AD is a tangent to the circle. BC is a diameter of the circle and ∠BCD = 30°. Find ∠DAB.

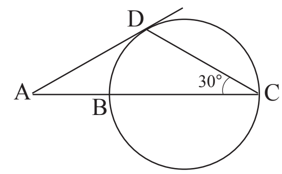


Fig. 9.85

10. In Fig. 9.86, AD is a tangent to the circle at D, ∠DAB = 28° and ∠ADC = 112°. Find the angle subtended at the centre of the circle by the chord BC.

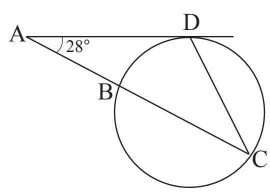


Fig. 9.86

11. Points A, B and C are on a circle such that ∠ABC = 108°. Find the angle between the tangents at A and C.

12. In Fig. 9.87, O is the centre of the circle. AB and CD are chords that meet at X. XT is a tangent to the circle.

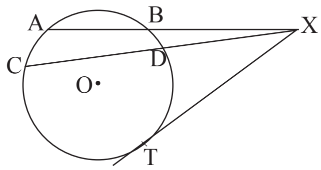


Fig. 9.87

Show that (a) $XT^2 = XA \cdot XB$

(b) $XT^2 = XC \cdot XD$.

9.2.7 Properties of chords

Perpendicular bisector of a chord

Activity 9.13

1. Draw a circle of any radius *r* cm, centre O. Draw a chord AB (not a diameter). From O draw a line perpendicular to AB, cutting AB at N. Measure AN and NB. What do you notice?
2. Draw a circle of any radius *r* cm, centre O. Draw a chord CD (not a diameter). Construct a perpendicular bisector of CD. What do you notice about the bisector and centre of the circle?
3. Repeat step 2 for any of the same circle.

Theorem 7

1. A perpendicular drawn from the centre of a circle to a chord bisects the chord.
2. A perpendicular bisector of a chord passes through the centre of the circle.

Example 9.20

Fig.9.89 shows a circle, centre O. M is the midpoint of the chord AB. Show that OM is perpendicular to AB.

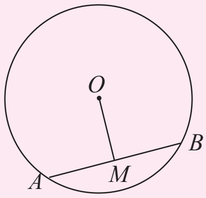


Fig. 9.89

Solution

Join OA and OB (Fig. 9.90)

In ∆s OMA and OMB, OA = OB (radii of the circle)

AM = MB (given that M is midpoint of AB)

OM (common to both triangles)

∴ ∆s OMA and OMB are congruent (sss)

∴ ∠OMA = ∠OMB.

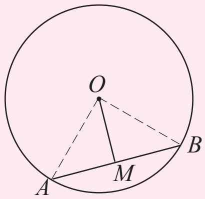


Fig. 9.90

But these are adjacent angles on a straight line

∴ ∠ OMA is a right angle, i.e. OM is perpendicular to AB.

Example 9.21

A chord AB of length 20 cm subtends an angle of 30° on the circumference of the major segment. Find:

(a) *the radius of the circle.*

(b) *the minor arc length AB (to 3 s.f.).*

Solution

(a) *∠AOB = 60° (angle subtended at the centre is twice angle subtended on the circumference by the same chord.*

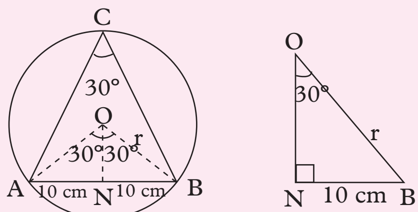


Fig. 9.91

$$Sin\ 30° = \frac{10}{r}$$

$$r = \frac{10}{sin\ 30^o}$$

$$= 20\ cm$$

(b) $$Arc\ AB = \frac{\theta}{360^o} \times 2\pi r$$

$$= \frac{60}{360^o} \times 2 \times \frac{22}{7} \times 20$$

$$= 21.0\ cm\ (to\ 3\ s.f.)$$

Using the properties noted above, we can calculate the length of a chord.

Example 9.22

A circle has a radius of 6 cm. A chord of the circle is 3.9 cm from the centre of the circle. Find the length of the chord.

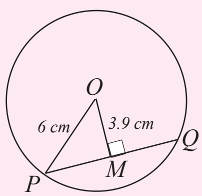


Fig. 9.92

Solution

In Fig. 9.88, let PQ be the chord whose length is required. O is the centre of the circle, radius 6 cm.

OM = 3.9 cm (the perpendicular distance from O to the chord)

In ΔPMO, $OP^2 = OM^2 + PM^2$ *(Pythagoras theorem)*

$$6^2 = 3.9^2 + PM^2$$

$$\Rightarrow PM^2 = 6^2 - 3.9^2$$

$$= 36 - 15.21 = 20.79$$

$$\therefore\ PM = 4.56\ cm$$

$$\therefore\ PQ = 2 \times PM = 2 \times 4.56\ cm$$

$$= 9.12\ cm.$$

Exercise 9.8

1. A chord of a circle of radius 13 cm is at a distance of 5 cm from the centre. Find the length of the chord.
2. A chord of a circle of radius 9 cm is at a distance of 4 cm from the centre. What is the length of the chord?
3. A chord of a circle is 10 cm long and is 12 cm from the centre of the circle. What is the radius of the circle?

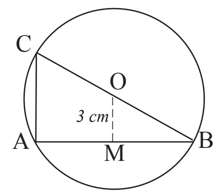


Fig. 9.93

4. A chord of a circle of radius 6 cm is 10 cm long. How far is the chord from the centre of the circle?
5. Fig. 9.94 shows a circle, radius 6 cm, centre O. Given that M is the midpoint of AB and is 3 cm from O, what is the length of AB?

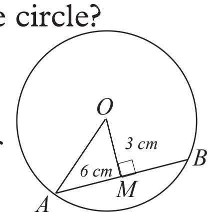


Fig. 9.94

6. A circle has a chord whose length is 9 cm. The chord is 4 cm from the centre of the circle. The same circle has a chord which is 3.5 cm from the centre, what is its length?
7. A chord of a circle is 12 cm long and 5 cm from the centre. What is the length of a chord which is 3 cm from the centre?
8. A chord of a circle of radius 5.5 cm subtends an angle of 42° at the centre. Find the difference in length between the chord and the minor arc.
9. P, Q and R are points on the circumference of a circle. If PQ = 12cm, PR = 12cm and QR = 8cm, what is the radius of the circle?

We can also extend the theorem of on perpendicular bisector of a chord to parallel and equal chords.

Activity 9.14

1. Draw a circle, centre O, radius 4 cm. Draw chords PQ and RS such that PQ//RS. Construct a perpendicular bisector of PQ and let it cut RS at point T. (Fig. 9.95).

Measure (i) $\angle OTR$ (ii) RT (iii) TS.

What do you notice?

What can you say about line OT?

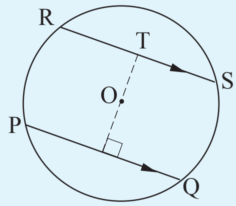


Fig. 9.95

2. Draw a circle, centre O, radius 4 cm. Draw two chords AB and CD such that AB = CD. Construct perpendicular bisectors OH and OK of AB and CD respectively (Fig. 9.96).

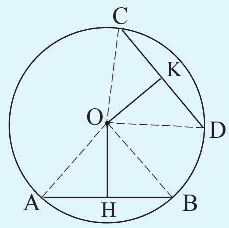


Fig. 9.96

Measure (i) OH and OK (ii) $\angle AOB$ and $\angle COD$. What do you notice?

In general:

1. If two chords of a circle are parallel, then the perpendicular bisector of one is also perpendicular bisector of the other.
2. The midpoints of parallel chords of a circle lie on a diameter.
3. If two chords of a circle are equal, then they are equidistant from the centre.
4. If two chords of a circle are equidistant from the centre, then their lengths are equal
5. If two chords of a circle are equal, then the angles they subtend at the centre are equal.
6. If two angles at the centre of a circle are equal, then they are subtended by equal chords.

Example 9.23

Two parallel chords of a circle are of lengths 8 cm and 12 cm respectively and are 10 cm apart. What is the diameter of the circle?

Solution

Consider Fig. 9.97.

In ΔPMO: $PO^2 = PM^2 + MO^2$ *(Pythagoras theorem).*

In ΔRNO:

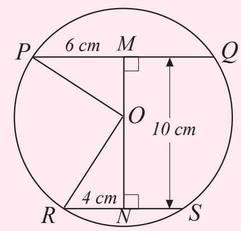


Fig. 9.97

$RO^2 = RN^2 + NO^2$ *(Pythagoras theorem).*

$PO = RO = r$ *(radius)*

Let $NO = x \Rightarrow MO = 10 - x$

Then $r^2 = 6^2 + (10 - x)^2$

and $r^2 = 4^2 + x^2$

Hence $16 + x^2 = 36 + (100 - 20x + x^2)$

$\Rightarrow 20x = 120$

i.e. $x = 6$

$\therefore r^2 = 4^2 + x^2 = 4^2 + 6^2$

$= 52$

$\Rightarrow r = 7.211$

$\therefore$ *Diameter* $= 14.42$ *cm*

Example 9.24

Fig. 9.98 shows a circle centre O, with two equal chords WX and YZ. Given that $\angle OSY = \angle OTW = 90°$*, show that* $OS = OT$.

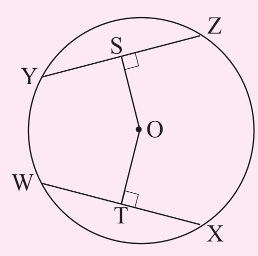


Fig. 9.98

Solution

Join OW and OY (Fig. 9.99). Since the perpendicular from the centre of a circle to a chord bisects the chord,

$WT = \frac{1}{2}WX$ and $YS = \frac{1}{2}YZ$

But $WX = YZ$ (given)

$\therefore WT = YS$.

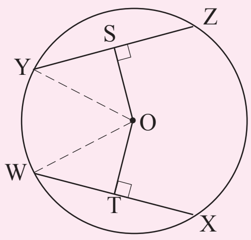


Fig. 9.99

In Δs OWT and OYS

$OW = OY$ (radii)

$WT = YS$ (Fig. 9.88)

$\angle OTW = \angle OSY$ (rt $\angle$s, given)

$\therefore \Delta OWT \equiv \Delta OYS$ (RHS)

$\therefore OS = OT$.

Note:

Equal chords of a circle are equidistant from the centre of circle.

Conversely, chords which are equidistant from the centre of a circle are equal.

Exercise 9.9

1. Two parallel chords of a circle are of lengths 3 cm and 5 cm respectively. What is the radius of the circle if the chords are;
   (a) 1 cm apart (b) 8 cm apart.
   (State your answers to 2 d.p.)
2. Two parallel chords of a circle are each 10 cm long. What is the perpendicular distance between the chords if the circle has a radius of 13 cm?
3. A circle of radius 2.5 cm has two parallel chords of lengths 3 cm and 4 cm. What is the distance between the chords? (Two possible answers).

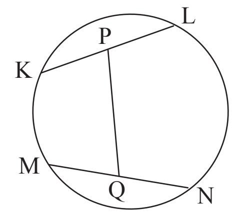


Fig. 9.100

4. PQ and RS are equal chords of a circle centre O. Show that $\angle POQ = \angle ROS$.
5. In Fig. 9.93 chords KL and MN are equal, and P and Q are their mid-points respectively. Show that $\angle KPQ = \angle MQP$.
6. A, B and C are points on the circumference of a circle.
   If AB = BC = CA = 6 cm:
   (a) what angle is subtended at the centre of the circle by AB?
   (b) what is the radius of the circle?
7. Points P, Q and R are points on the circumference of a circle.
   If PQ = PR = 13 cm and QR = 10 cm, what is the radius of the circle?
8. Two parallel chords of a circle are of lengths 12 cm and 16 cm. If the radius of the circle is 10 cm, what are the two possible perpendicular distances between the chords?
9. KL is a chord of length 15 cm and is 10 cm from the centre of a circle. What is the length of a chord which is 12 cm from the centre?

Unit Summary

- Angles in the same segment i.e. subtended by the same arc, are equal. In Fig. 9.101, $p = q$. $x = 2p = 2q$

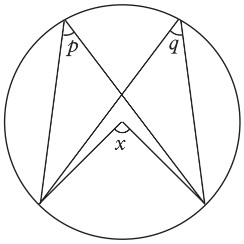


Fig. 9.101

- The angle subtended at the centre of a circle by an arc is double that subtended by the same arc on the remaining part of the circumference. In Fig. 9.101, $x = 2p$ or $x = 2q$.
- The angle subtended by the diameter on the circumference (i.e. the angle in a semicircle) is a right angle. In Fig. 9.102, $x = y = 90°$.

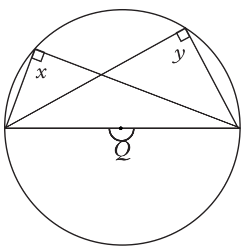


Fig. 9.102

$$x = y = \frac{1}{2}Q = \frac{180°}{2} = 90°$$

- Opposite angles of a cyclic quadrilateral (a quadrilateral with its four vertices lying on the circumference of a circle) are supplementary (i.e. they add up to 180°). If one side of a cyclic quadrilateral is produced, the exterior angle thus formed equals the interior opposite angle (Fig. 9.103).

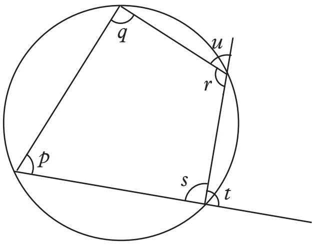


Fig. 9.103

$$p + r = 180°$$
$$q + s = 180°$$
$$t = q$$
$$u = p$$

- The perpendicular bisector of a chord passes through the centre of the circle. It also bisects the angle that the chord subtends at the centre.
- Equal chords of a circle are equidistant from the centre of the circle.
- If two or more chords of a circle are parallel, then the perpendicular bisector of one bisects the others.
- The midpoints of parallel chords lie on a diameter.
- Equal chords subtend equal angles at the centre.
- The tangent to a circle at a point is perpendicular to the radius of the circle at that point.
- Two tangents to a circle, drawn from the same point, have the same length (Fig. 9.104).

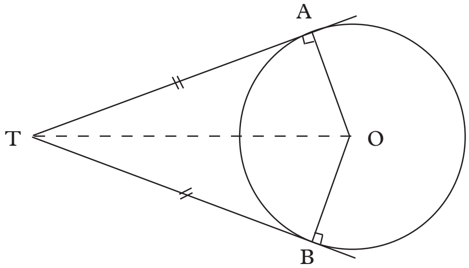


Fig. 9.104

- The angle formed by a tangent of a circle and a chord drawn from the point of contact is equal to the angle that the chord subtends in the alternate segment (Fig. 9.105).

(a)

(b)

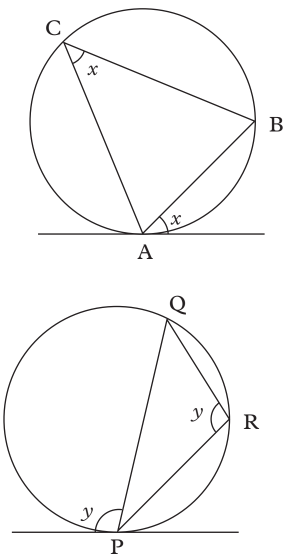


Fig. 9.105

Unit 9 Test

1. In Fig. 9.106, find the angles marked with letters a, b, c, d and e.

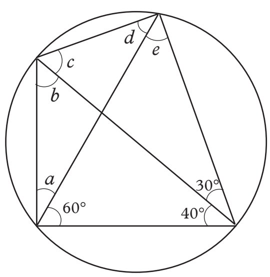


Fig. 9.106

2. In Fig. 9.107, find the angles marked by the letters a, b, c and d.

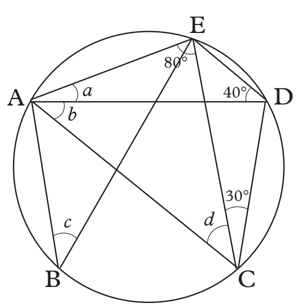


Fig. 9.107

3. In Fig. 9.108, AB is a diameter. Find the sizes of the angles marked x and y.

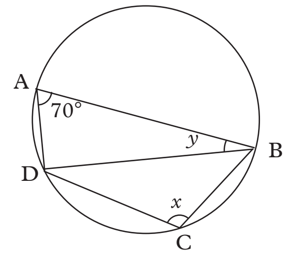


Fig. 9.108

4. Fig 9.109 shows a rectangle ABCD which is inscribed in a circle, centre O and radius 10 cm. Given tha

AB = 16 cm, calculate

(taking $\pi = 3.142$)

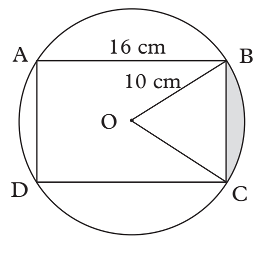


Fig. 9.109

(a) the width BC of the rectangle,

(b) the central angle BOC,

(c) the length of the minor arc BC,

(d) the shaded area.

5. In Fig. 9.110, PQ and PT are tangents to the circle. Given that $\angle OQU = 28°$ and OQS = 15°, find the sizes of angles PQU, QSU, QUS and QRS.

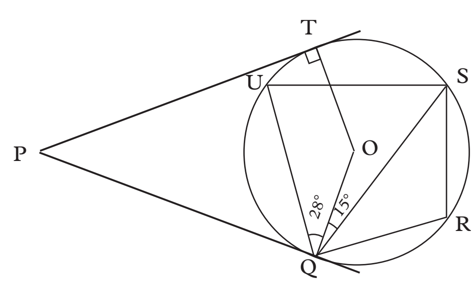


Fig. 9.110

6. A chord of a circle of length 15 cm subtends an angle of 120° at the centre. Calculate the radius of the circle and the length of the minor arc.

7. Find the marked angles in the quadrilaterals in Fig. 9.111 (a) and (b).

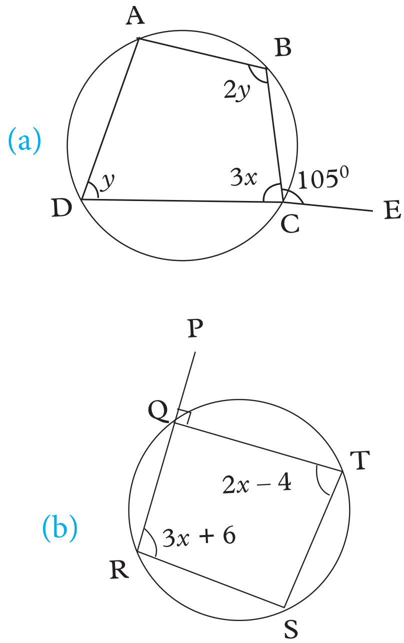


Fig. 9.111

8. The angles of a cyclic quadrilateral are $6x$, $3x$, x +y and $3x + 4y$ in that order. Determine the values of x and y, and hence the sizes of the angles of the quadrilateral.

9. Find the size of the angle marked x in Fig. 9.112.

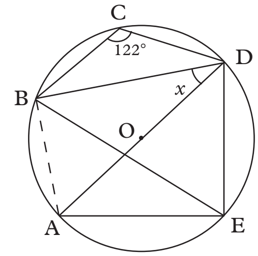


Fig. 9.112

10. In Fig. 9.113, BOE is a diameter, ∠CAE = 45°, ∠BEA = 50°, ∠BEC = 25°, ∠DCE = 20° and ∠DEF = 130°. Find:

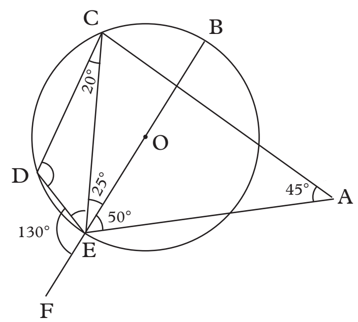


Fig. 9.113

(a) ∠CED (b) ∠DCB

(c) ∠ACE (d) ∠CDE

(e) ∠CBE

10 COLLINEAR POINTS AND ORTHOGONAL VECTORS

Key unit competence: By the end of this unit, learners should be able to apply properties of collinearity and orthogonarity to solve problems involving vectors.

Unit outline

- Collinear points
- Orthogonal vectors

Introduction

Unit Focus Activity

1. Consider the points
   (a) P(–1, 2) Q(1, 4) and R(3, 6)
   (b) L(3, –2) Q(1, 0) and N(0, 3)
      (i) Without plotting determine whether the three points in each are collinear (lie on the same straight line)
      (ii) State one advantage of three or more objects lying on the same straight line.
2. Consider the column vectors.
   (a) $AB = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $CD = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$
   (b) $PQ = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, $RS = \begin{pmatrix} 12 \\ -8 \end{pmatrix}$
   Determine whether the two pairs of vectors are orthogonal in each case.

10.1 Collinear points

10.1.1 Definition of colinearity

Activity 10.1

1. You are given three points A (2, 2), B (3, 3) and C (6, 6).
   (a) Plot the points ABC on a Cartesian plane.
   (b) Join the points A, B and C using a ruler.
2. Given the points P (–2, 3), Q (3, 3) and R (1, –5).
   (a) Plot the points P, Q and R on a Cartesian plane
   (b) Join the points P, Q and R using a ruler
3. What observation do you make in the plotted points A, B, C and P, Q, R?
4. How would you have determined what you observed in step 3 without drawing?

Three or more points are said to be collinear if they lie on a single straight line. For example the points A, B and C in Fig. 10.1 are collinear because they lie on a single straight line.

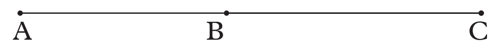


Fig. 10.1

Points E, F and G in Fig.10.2 below are not collinear because they don't lie on the same straight line.

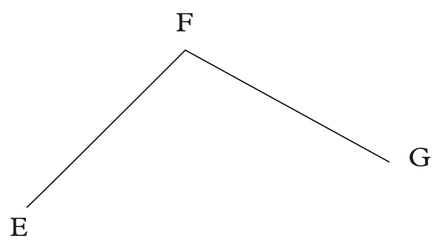
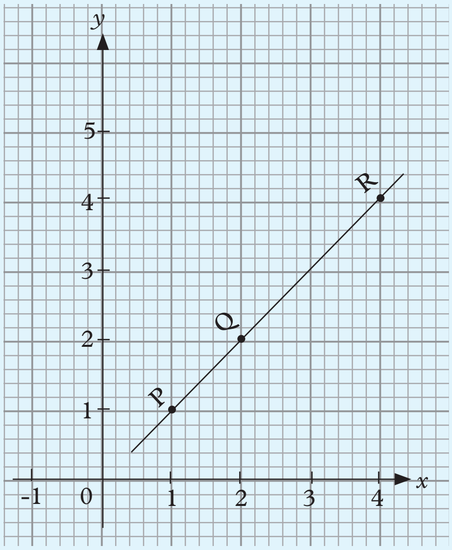


Fig. 10.2

10.1.2 Verifying collinearlity of points using vector laws

Activity 10.2

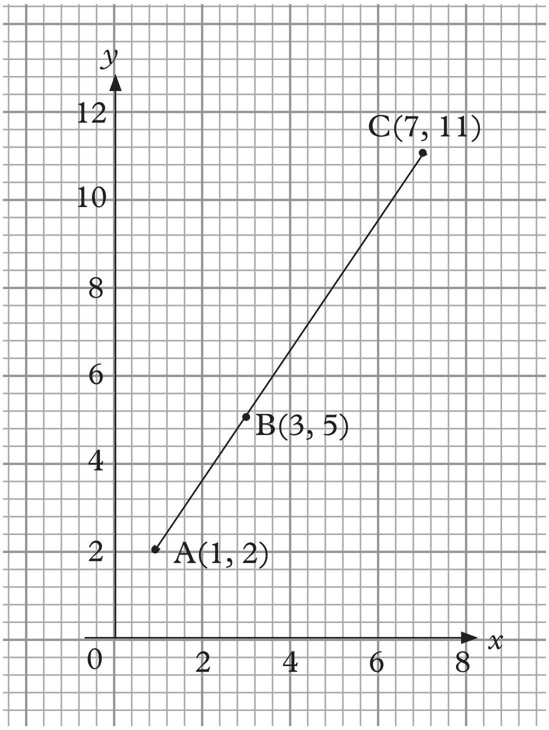
Observe the straight line PQR in graph 10.1 below.



Graph 10.1

1. Determine the column vectors PQ and QR.
2. Express vector PQ in terms of QR.
3. Since points P, Q and R are collinear, state one condition for collinearlity guided by relationship of vectors PQ and QR that you obtained in step 3.

Consider another straight line ABC in graph 10.2 below with point B located between A and C as shown.



Graph 10.2

If ***AB*** = k***BC*** where ***k*** is a scalar, then ***AB*** is parallel to ***BC***.

Since B is a common point between vectors ***AB*** and ***BC***, then A, B and C lie on a straight line, i.e. the points A, B and C are **collinear**.

Example 10.1

Show that the points A (0, -2), B (2, 4) and C (-1, -5) are collinear.

Solution

AB = *k**BC***.

OB – ***OA*** = *k*(***OC*** – ***OB***)

$\begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}0\\-2\end{pmatrix} = k\begin{pmatrix}-1\\-5\end{pmatrix} - \begin{pmatrix}2\\4\end{pmatrix}$

$\begin{pmatrix}2\\6\end{pmatrix} = k\begin{pmatrix}-3\\-9\end{pmatrix}$

(i) $2 = 3k \Rightarrow k = \frac{-2}{3}$

(ii) $6 = -9k \Rightarrow k = \frac{-6}{9} = \frac{-2}{3}$

Since the value of k is the same for the two cases (i) and (ii), i.e. $\mathbf{AB} = \frac{-2}{3}\mathbf{BC}$, *and* **B** *is a common point of two vectors* **AB** *and* **BC**, *then points* **A, B** *and* **C** *are collinear.*

Example 10.2

For what value of k are the following points collinear? A(1, 5), B(k, 1) and C(11, 7)

Solution:

Let the points be A, B and C. For the points to be collinear, B can be a common point and therefore we get; $\mathbf{AB} = a\mathbf{BC}$ *where a is a scalar*

$$\mathbf{AB} = \mathbf{OB} - \mathbf{OA} \Rightarrow \begin{pmatrix} k \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} k-1 \\ -4 \end{pmatrix}$$

$$\mathbf{BC} = \mathbf{OC} - \mathbf{OB} \Rightarrow \begin{pmatrix} 11 \\ 7 \end{pmatrix} - \begin{pmatrix} k \\ 1 \end{pmatrix} = \begin{pmatrix} 11-k \\ 6 \end{pmatrix}$$

Hence $\begin{pmatrix} k-1 \\ -4 \end{pmatrix} = a\begin{pmatrix} 11-k \\ 6 \end{pmatrix}$

We get $-4 = 6a \Rightarrow a = \frac{-4}{6} = \frac{-2}{3}$

and $k - 1 = a(11 - k)$

Substituting the value of,

$k - 1 = \frac{-2}{3}(11 - k)$

$\Rightarrow 3(k - 1) = -2(11 - k)$

$\Rightarrow 3k - 3 = -22 + 2k$

$\Rightarrow 3k - 2k = -22 + 3$

Hence $k = -19$

Exercise 10.1

1. Verify whether the following points are collinear.
   (a) A(2, 3), B(9, 8) and (5, 4)
   (b) P(–1, 1), Q(5, 1) and R(2, 0)
   (c) X(–2, 3), Y(7, 0) and Z(1, 2)
   (d) R(1, 2), S(4, 0) and T(–2, 4)

2. (a) The following points are collinear, find the values of unknown in each case.
   (i) (1, 0), (*k*, 3), (4, 5)
   (ii) (5, 1), (*x*, 2), (*x*, 7)
   (iii) (2, 4), (1, *k*), (6, *k*)

   (b) Plot all the points in (a) on a Cartesian plane. What do you observe?

10.1.3 Applications of collinearlity in proportional division of lines

Activity 10.3

Consider line AB in Fig.10.3,

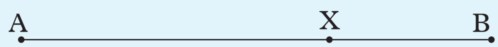


Fig.10.3

Points **A**, **X** and **B** are collinear. X divides line **AB** in a ratio such that **AX: XB** = 2:1

(a) Express **AX** in terms of **AB**

(b) Express **XB** in terms of **AB**

(c) Express **AX** in terms of **XB**

Consider the following cases;

(a)

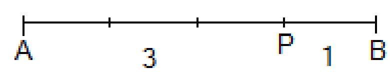


Fig.10.4

AP:PB = 3:1, then we say that P divides **AB** internally in the ratio 3: 1. (P lies between A and B).

Note: The direction is important as P divides **BA** in the ratio 1: 3.

$\mathbf{AP} = \frac{3}{4}\mathbf{AB}$ and $\mathbf{PB} = \frac{1}{4}\mathbf{AB}$

(b)

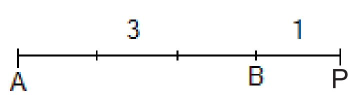


Fig.10.5

AP*: *PB = 4: –1.

The negative sign is required as ***PB*** is in the opposite direction to ***AB***. We say that ***P*** divides ***AB*** externally in the ratio 4: –1, or ***P*** divides ***BA*** in the ratio –1 : 4

Whenever *P* is outside ***AB*** (on either side), ***AP*: *PB*** will be negative, and we say that ***P*** divides ***AB*** externally.

Example 10.3

In Fig.10.5 OCD, OC = c and OD = d. C, M and D lie on the same straight line.

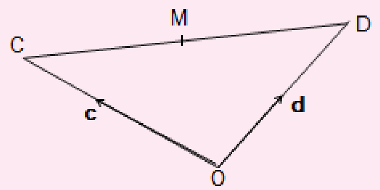


Fig.10.6

(a) *Express* ***CD*** *in terms of vectors* ***c*** *and* ***d***.

(b) *M is the midpoint of* ***CD***. *What is* ***CM*** *in terms of* ***c*** *and* ***d***?

(c) *Using your answers to (a) and (b), find* ***OM*** *in terms of* ***c*** *and* ***d***.

Solution

(a) $\boldsymbol{CD} = \boldsymbol{CO} + \boldsymbol{OD} = -\boldsymbol{OC} + \boldsymbol{OD}$

$= -\mathbf{c} + \mathbf{d}$ or $\mathbf{d} - \mathbf{c}$

Therefore, $\boldsymbol{CD} = \mathbf{d} - \mathbf{c}$.

(b) $\boldsymbol{CM} = \frac{1}{2}\,\boldsymbol{CD}$

$= \frac{1}{2}\mathbf{CD}$

(c) $\boldsymbol{OM} = \boldsymbol{OC} + \boldsymbol{CM}$

$= c + \frac{1}{2}(\mathbf{d} - \mathbf{c})$

$= \frac{1}{2}\mathbf{c} + \frac{1}{2}\mathbf{d}$

$= \frac{1}{2}(\mathbf{c} + \mathbf{d})$

Example 10.4

In the Fig.10.6 below, A, B and P are collinear points. P divides the line AB in the ratio ***AP*: *PB*** *= 7: 3. If* **OA** *=* ***a*** *and* **OB** *=* ***b***, *express OP in terms of* ***a*** *and* ***b***.

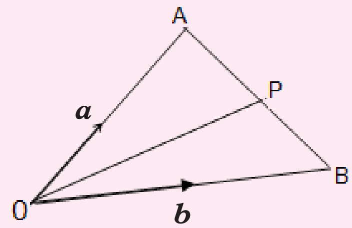


Fig.10.7

Solution

In triangle ***OAB***, $\boldsymbol{OA} + \boldsymbol{AB} = \boldsymbol{OB}$

$\boldsymbol{a} + \boldsymbol{AB} = \boldsymbol{b}$

$\boldsymbol{AB} = \boldsymbol{b} - \boldsymbol{a}$.

Along ***AB***, $\boldsymbol{AP} = \frac{7}{10}\boldsymbol{AB} = \frac{7}{10}(\boldsymbol{b} - \boldsymbol{a})$

$\boldsymbol{OP} = \boldsymbol{OA} + \boldsymbol{AP} = \boldsymbol{a} + \frac{7}{10}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{3}{10}\boldsymbol{a} + \frac{7}{10}\boldsymbol{b}$.

Example 10.5

In the Fig. 10.7 below, **OA** *= a and* **OB** *= b.*

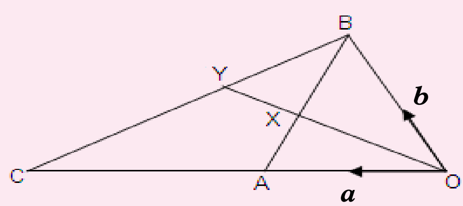


Fig.10.8

(a) *Express* **BA** *in terms of* ***a*** *and* ***b***.

(b) *If X is the mid-point of* **BA**, *show that* **OX** $= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$.

(c) *Given that* **OC** *= 3****a***, *express BC in terms of* ***a*** *and* ***b***.

(d) *Given that* **BY** *= m***BC**, *express* **OY** *in terms of a,b,m.*

Solution

(a) *In triangle OAB,*

$\mathbf{OB} + \mathbf{BA} = \mathbf{OA}$

$\mathbf{b} + \mathbf{BA} = \mathbf{a}.$

$\mathbf{BA} = \mathbf{a} - \mathbf{b}.$

(b) *In triangle* **OBX**, $\mathbf{BX} = \frac{1}{2}\mathbf{BA}$

$\mathbf{BX} = \frac{1}{2}(\mathbf{a} - \mathbf{b})$

$\mathbf{OX} = \mathbf{OB} + \mathbf{BX} = \mathbf{b} + \frac{1}{2}(\mathbf{a} - \mathbf{b}).$

$\mathbf{OX} = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$

(c) *In triangle* **OBC**, $\mathbf{BC} = \mathbf{BO} + \mathbf{OC}$

$\mathbf{BC} = -\mathbf{b} + 3\mathbf{a}.$

$\mathbf{BC} = 3\mathbf{a} - \mathbf{b}.$

(d) *In triangle* **OBY**, $\mathbf{OY} = \mathbf{OB} + \mathbf{BY}$

$\mathbf{OY} = \mathbf{b} + m(3\mathbf{a} - \mathbf{b})$

$\mathbf{OY} = 3m\mathbf{a} + (1 - m)\mathbf{b}$

Exercise 10.2

1. In the diagram in Fig.10.9, $\mathbf{OD} = 2\mathbf{OA}$, $\mathbf{OE} = 4\mathbf{OB}$, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$. Points ***B***, ***A*** and ***C*** are collinear.

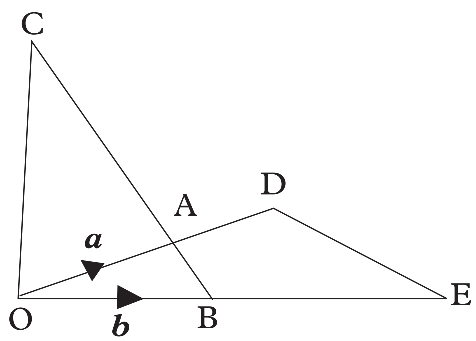


Fig.10.9

   (a) Express the following vectors in terms of ***a*** and ***b***. ***OD***, ***OE***, ***BA*** and ***ED***.

   (b) Given that $\mathbf{BC} = 3\mathbf{BA}$, express:

   (i) ***OC***

   (ii) ***EC***, in terms of ***a*** and ***b***.

   (c) Hence show that the points ***E***, ***D*** and ***C*** lie on a straight line.

2. In the diagram in Fig 10.10, $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and ***M*** is the mid-point of ***AB***. Points ***A***, ***M*** and ***B*** are collinear.

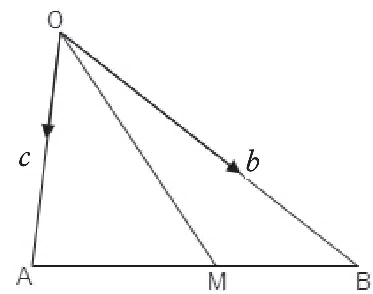


Fig.10.10

   Find ***OM*** in terms of *a* and *b*.

3. In Figure 10.11, OABC is a parallelogram, ***M*** is the mid-point of ***OA*** and $\mathbf{AX} = 7\mathbf{AC}$, $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OC} = \mathbf{c}$. Points A, Y and B, M, X and ***Y***, ***A***, ***X*** and ***C*** are collinear.

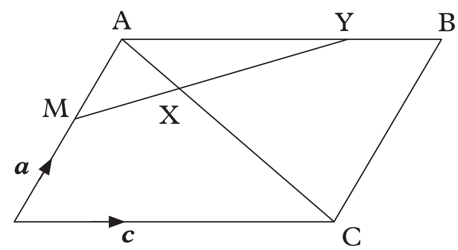


Fig.10.11

   (a) Express the following in terms of ***a*** and ***c***.

   (i) ***MA*** (ii) ***AB***

   (iii) ***AC*** (iv) ***AX***

   (b) Using triangle ***MAX***, express ***MX*** in terms of ***a*** and ***c***.

   (c) If $\mathbf{AY} = p\mathbf{AB}$, use triangle ***MAY*** to express ***MY*** in terms of ***a***, ***c*** and ***p***.

(d) Also if $\boldsymbol{MY} = q\boldsymbol{MX}$, use the result in (b) to express $\boldsymbol{MY}$ in terms of a, c and q.

(e) Hence find p and q and the ratio $\boldsymbol{AY}: \boldsymbol{YB}$

4. ORST is a parallelogram and S, P and T are collinear in Fig.10.12.

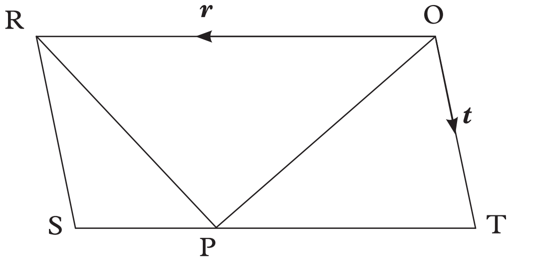


Fig.10.12

If $\boldsymbol{ST} = 4\boldsymbol{SP}$, express the following in terms of **r** and or **t**.

(a) $\boldsymbol{RS}$ (b) $\boldsymbol{ST}$ (c) $\boldsymbol{SP}$

(d) $\boldsymbol{RP}$ (e) $\boldsymbol{OP}$

5. In Fig. 10.13, PQR is a triangle, $\mathbf{PQ} = \boldsymbol{a}$, $\boldsymbol{PR} = \boldsymbol{b}$ and S is the mid-point of $\boldsymbol{RQ}$.

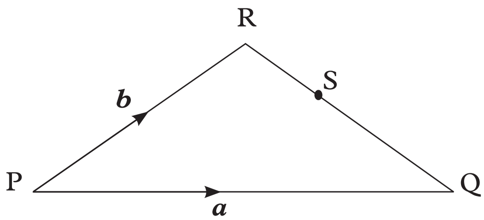


Fig.10.13

Express the following in terms of $\boldsymbol{a}$ and or $\boldsymbol{b}$.

(a) $\boldsymbol{QR}$ (b) $\boldsymbol{PS}$ (c) $\boldsymbol{QS}$

6. (a) In Fig.10.14: $\boldsymbol{OY} = 2\mathbf{OB}$,

$\boldsymbol{OX} = \frac{5}{2}\boldsymbol{OA}$, $\boldsymbol{OA} = \boldsymbol{a}$ and $\boldsymbol{OB} = \boldsymbol{b}$,

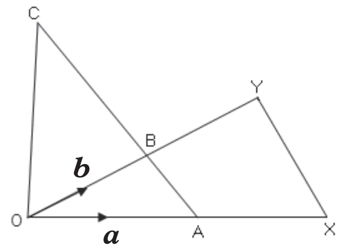


Fig.10.14

Express the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(a) $\boldsymbol{OY}$, $\boldsymbol{OX}$, $\boldsymbol{AB}$, and $\boldsymbol{XY}$.

(b) Given that $\boldsymbol{AC} = 6\boldsymbol{AB}$, express $\boldsymbol{OC}$ and $\boldsymbol{XC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(c) Use the results for $\boldsymbol{XY}$ and $\boldsymbol{XC}$ to show that points X, Y and C lie on a straight line.

7. In the diagram below $\boldsymbol{OA} = \boldsymbol{a}$ and $\boldsymbol{OB} = \boldsymbol{b}$, M is the mid-point of $\boldsymbol{OA}$ and P lies on $\boldsymbol{AB}$ such that $\boldsymbol{AP} = 3\,\boldsymbol{AB}$.

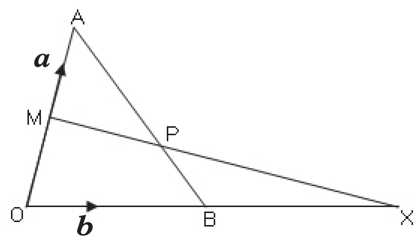


Fig.10.15

(a) Express the following in terms of a and b: $\boldsymbol{AB}$, $\boldsymbol{AP}$, $\boldsymbol{MA}$ and $\boldsymbol{MP}$

(b) If X lies on $\boldsymbol{OB}$ produced such that $\boldsymbol{B} = \boldsymbol{BX}$, express $\boldsymbol{MX}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(c) Show that MPX is a straight line.

10.2 Orthogonal vectors

Activity 10.4

1. Research from reference books or internet how to determine the product of two column vectors.

2. You are given the vectors

   $\boldsymbol{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

   a) Find the product of **a** and **b**.

   b) Plot the vectors in a Cartesian plane. What is the angle between them?

3. You are given the vectors

   $\boldsymbol{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

   (a) Find the product of p and q.

   (b) Plot the vectors on a Cartesian plane. What is the angle between them?

4. Based on your results in 2 and 3, what is the relationship between two perpendicular vectors and the product of the vectors?

Two vectors are said to be perpendicular if the angle between them is 90° (i.e. if they form a right angle).

The product of two vectors must be zero for the angle between them to be 90°.

Consider two vectors $\boldsymbol{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

Two vectors are said to be orthogonal if the sum of product of tops and the product of bottoms is zero.

If vectors **a** and **b** are orthogonal, then $x_1x_2 + y_1y_2 = 0$.

Note: For $a = \begin{pmatrix} x \\ y \end{pmatrix}$ ... top ... bottom

Example 10.7

(a) Show that the vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\boldsymbol{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ *are orthogonal vectors*

Solution

(a) Vectors **a** *and* **b** *are orthogonal vectors if* $x_1x_2 + y_1y_2 = 0$

$$= (2 \times -1) + (1 \times 2)$$

$$= -2 + 2 = 0.$$

(b) These vectors are orthogonal.

Example 10.8

Find the value of t such that $\boldsymbol{a} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ *and* $\boldsymbol{b} = \begin{pmatrix} 2 \\ t \end{pmatrix}$ *are orthogonal.*

Solution

If **a** *and* **b** *are orthogonal, then*

$$(-1 \times 2) + (5 \times t) = 0$$

$$-2 + 5t = 0$$

$$5t = 2$$

$$t = \frac{2}{5}$$

Example 10.9

Given the vectors $\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$,

A (3,2), B (7,6, C(2,2), D(5,5)

$\boldsymbol{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ *and* $\boldsymbol{d} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

(a) Find vectors ***AB*** *and* ***CD.***

(b) Show that ***AB*** *is parallel to* ***CD.***

Solution

(a) $\boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA}$

$$\boldsymbol{AB} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$\boldsymbol{CD} = \boldsymbol{OD} - \boldsymbol{OC}$

$$\boldsymbol{CD} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

(b) $\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \frac{4}{3}\begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$\boldsymbol{AB} = \frac{4}{3}\boldsymbol{CD}$. *Hence parallel.*

$\boldsymbol{AB} = \frac{4}{3}\boldsymbol{CD}$

Exercise 10.3

1. Show whether the following vectors are orthogonal.
   (a) $\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}$
   (b) $\boldsymbol{u} = \begin{pmatrix} -7 \\ -3 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
   (c) $\boldsymbol{u} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$
   (d) $\boldsymbol{u} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} -4 \\ 20 \end{pmatrix}$
2. Consider points A(5, 3), B(2, –1) and C(7, –3,). Find:
   (i) $\boldsymbol{BA}$ and $\boldsymbol{BC}$
   (ii) Show whether $\boldsymbol{BA}$ and $\boldsymbol{BC}$ are orthogonal. Give reasons.
3. Let $a = \begin{pmatrix} k \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ be two vectors. Find the values of k if; a and b are perpendicular
4. Are the following given vectors perpendicular? Write true/false for the result obtained.
   (a) $\boldsymbol{a} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$
   (b) $\boldsymbol{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
   (c) $\boldsymbol{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
   (d) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
   (e) $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$
   (f) $\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
   (g) $\boldsymbol{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
5. The co-ordinates of P, Q, R and S are (0, 1), (7, 8), (1, –1) and (9, 7).
   (a) Find $\boldsymbol{PQ}$ and $\boldsymbol{RS}$.
   (b) What is the relationship between $\boldsymbol{PQ}$ and $\boldsymbol{RS}$?

Unit Summary

- **Collinearlity:** Three or more points are said to be collinear if they lie on a single straight line.

  For example if A, B and C are three points on the same straight line ABC, then $\boldsymbol{AB} = k\boldsymbol{BC}$ where **B** becomes a common point.
- **Collinear points** find their applications in proportional line divisions.

  For example, consider the following cases;

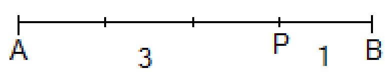


Fig. 10.16

AP: PB = 3: 1, then we say that P divides AB internally in the ratio 3: 1. (P lies between A and B).

Note: The direction is important as P divides BA in the ratio 1: 3.

$\boldsymbol{AP} = \frac{3}{4}\boldsymbol{AB}$ and $\boldsymbol{PB} = \frac{1}{4}\boldsymbol{AB}$
- **Orthogonal vectors:** Two vectors are said to be perpendicular if the angle between them is 90° (i.e. if they form a right angle).

The product of two vectors must be zero for the angle between them to be 90°.

Consider two vectors $\boldsymbol{a} = \binom{x_1}{y_1}$ and $\boldsymbol{b} = \binom{x_2}{y_2}$ the sum of the product of tops and bottoms is defined by the formula;

$= x_1x_2 + y_1y_2$

If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, then $x_1x_2 + y_1y_2 = 0$.

Unit 10 Test

1. Find the value of t such that $\boldsymbol{a} = \binom{-1}{5}$ and $\boldsymbol{b} = \binom{2}{t}$ are orthogonal.
2. In the diagram, $\boldsymbol{OA} = \boldsymbol{a}$, $\boldsymbol{OB} = \boldsymbol{b}$ and M is the mid-point of $\boldsymbol{AB}$.

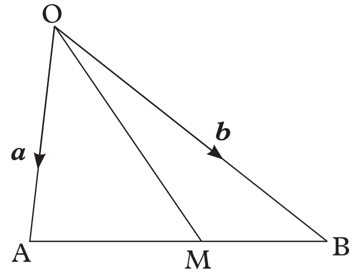


Fig. 10.17

Find $\boldsymbol{OM}$ in terms of a and b.

3. In the diagram $\boldsymbol{OA} = \boldsymbol{a}$, and $\boldsymbol{OB} = \boldsymbol{b}$. The points P and Q are such that $\boldsymbol{AP} = \frac{1}{3}\boldsymbol{AB}$ and $\boldsymbol{OQ} = \frac{1}{3}\boldsymbol{OA}$.

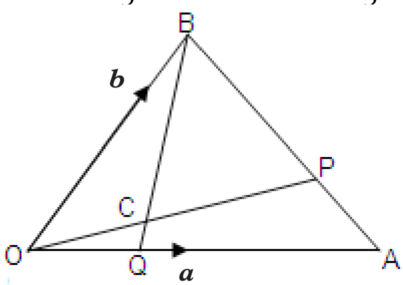


Fig. 10.18

Express $\boldsymbol{OP}$ and $\boldsymbol{BQ}$ in terms of a and b.

If $\boldsymbol{OC} = h\boldsymbol{OP}$ and $\boldsymbol{BC} = k\boldsymbol{BQ}$, express $\boldsymbol{OC}$ in two different ways and hence deduce

(a) The values of h and k.

(b) The vector $\boldsymbol{OC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ only.

(c) The ratio in which **C** divides $\boldsymbol{BQ}$.

4. In triangle OXY, $\boldsymbol{OX} = x$ and $\boldsymbol{OY} = \mathbf{y}$. The point A lies on OY such that $\boldsymbol{OA} : \boldsymbol{AY} = 1:3$. The point B lies on $\boldsymbol{XY}$ such that $\boldsymbol{XB} : \boldsymbol{BY} = 2 : 3$. Find in terms of **x** and **y** the vectors $\boldsymbol{OA}$, $\boldsymbol{XY}$, $\boldsymbol{XB}$, $\boldsymbol{OB}$ and $\boldsymbol{AX}$. The lines OB and AX intersect at point C. Given that $\boldsymbol{OC} = h\ \boldsymbol{OB}$ and $\boldsymbol{AC} = k\ \boldsymbol{AX}$, express the position vector of **C** in terms of:

(a) $\boldsymbol{h}, \boldsymbol{x}$ and $\boldsymbol{y}$,

(b) $\boldsymbol{k}, \boldsymbol{x}$ and $\boldsymbol{y}$.

Hence find the values of h and k and determine the ratios $\boldsymbol{OC}: \boldsymbol{CB}$ and $\boldsymbol{AC} : \boldsymbol{CX}$.

5. In triangle OPQ; $\boldsymbol{OP} = p$, and $\boldsymbol{OQ} = q$. **A** and **B** are points on $\boldsymbol{OQ}$ and $\boldsymbol{OP}$ respectively, such that $\boldsymbol{OA}: \boldsymbol{AQ} = 3:1$ and $OB = \frac{1}{4}\boldsymbol{OP}$. $\boldsymbol{AP}$ and $\boldsymbol{BQ}$ intersect at **C**.

(a) Show that $\boldsymbol{OC} = \frac{1}{13}(p + 9q)$.

(b) Determine the ratios $\boldsymbol{AC}: \boldsymbol{CP}$ and $\boldsymbol{BC} : \boldsymbol{CQ}$.

6. (a) Find the sum of product of the bottoms and tops of vectors $\boldsymbol{a} = \binom{2}{1}$ and $\boldsymbol{b} = \binom{-1}{2}$.

(b) What is your conclusion about the result in (a) above?

7. Show that the points A (0, –**2**), B (2, **4**) and C (–1, –5) are collinear.
8. Are the following vectors orthogonal? Write True or False.

(a) $\boldsymbol{a} = (4, -12)$ and $\boldsymbol{b} = (1, -11)$.

(b) $\boldsymbol{a}$ = (–9, –4) and $\boldsymbol{b}$ = (2, 4).

(c) $\boldsymbol{a}$ = (–10, –20) and $\boldsymbol{b}$ = (13, 6).

9. Find the value of $\boldsymbol{n}$ for which the vectors $\boldsymbol{a}$ = (–15, $\boldsymbol{n}$) and $\boldsymbol{b}$ = (2, 19) are orthogonal.

10. What is the value of $\boldsymbol{n}$ for which the vectors $\boldsymbol{a}$ = (13, 19) and $\boldsymbol{b}$ = ($\boldsymbol{n}$, –6) are orthogonal?

11. Find the value of $\boldsymbol{x}$ at which the vectors $\boldsymbol{a}$ = (9, x) and $\boldsymbol{b}$ = (4, 13) are orthogonal.

12. Find the value of $\boldsymbol{p}$ for which vectors $\boldsymbol{a}$ = ($\boldsymbol{p}$, –19) and $\boldsymbol{b}$ = (17, 14) are orthogonal?

13. What is the value of $\boldsymbol{k}$ for which the vectors $\boldsymbol{a}$ = (18, k) and $\boldsymbol{b}$ = (8, 8) are orthogonal.

14. Find the value of $\boldsymbol{n}$ at which the vectors $\boldsymbol{a}$ = (0, $\boldsymbol{n}$) and $\boldsymbol{b}$ = (–1, –15) are orthogonal.

11 ENLARGEMENT AND SIMILARITY IN 2D

Key unit competence

By the end of this unit, the learner should be able to solve shape problems about enlargement and similarities in 2D.

Unit Outline

- Definition and properties of similarity.
- Similar polygons
- Similar triangles
- Finding length of sides of shapes using similarity and Thales theorem
- Definition of enlargement
- Properties of similarity
- Determining linear scale factor of enlargement
- Determining the centre of enlargement
- Enlargement in the Cartesian plane
- Areas and volumes of similar shapes and objects respectively
- Composite and inverse enlargement

Introduction

Unit Focus Activity

One of the coffee processing and marketing companies in Rwanda packages its coffee sachets in small cartons for domestic use and export.

Suppose that each carton measures 24 cm long, 12 cm wide and 16 cm high.

The company is planning to make a very big model of the carton, 1.8 m long to be mounted on the roadside of a busy highway for advertisement.

The company manager was told that you learn Mathematics in school. So, he has invited you to advice them on all the dimensions (measurements) they should use in making the model, so that it will have exactly the same shape as the carton.

Fig. 11.1 Packet and its model

1. Identify and state two concepts in Mathematics that will help you determine the correct dimensions of the model.
2. Use these concepts to determine the correct dimensions of the model.
3. Compare your dimensions with those obtained by other classmates in class discussion.
4. How can you advice the company on the disposal of used-up cartons.

Drink Rwanda grown coffee and use other Rwanda made products to grow our economy and support our farmers and producers

11.1 Similarity

11.1.1 Similar triangles

Activity 11.1

1. Use a mathematics dictionary, reference books or Internet to find out the definition of similarity
2. Draw a triangle with sides 4.6 cm, 3.0 cm and 3.4 cm, measure its angles.
3. Draw another triangle PQR whose lengths are $1\frac{1}{2}$ times as long as the first triangle of ABC.
4. Measure the angles of the two triangles. What do you notice?
5. Compare the shapes of the two triangles. What do you notice. Explain your observation.

Activity 11.2

1. Draw accurately two triangles ABC and PQR such that ΔPQR is smaller than ΔABC (Fig. 11.12).

   Let $\angle A = \angle P = 55^o$, $\angle B = \angle Q = 75^o$ and $\angle C = \angle R = 50^o$, AB = 3.6 cm and PQ = 1.8 cm.

(a)

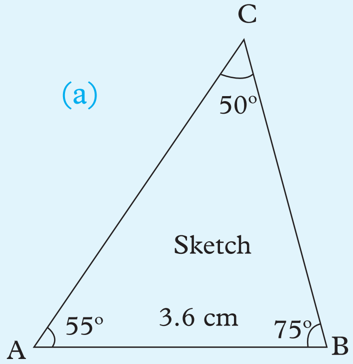


Fig. 11.2

Mesure and record the lengths:

AB = _______ cm

BC = _______ cm

AC = _______ cm

(b)

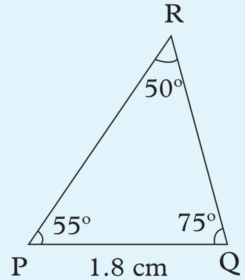


Fig. 11.3

2. Measure and record the lengths AC, BC, PR and QR in centimetres.
3. Find the ratios $\frac{AB}{PQ}$, $\frac{BC}{QR}$ and $\frac{AC}{PR}$. What do you notice?
4. Are Δs ABC, PQR similar?

Activity 11.3

1. Observe triangles ABC and DEF (see Fig. 11.4) and draw them accurately such that $\angle A = \angle D$, $\angle C = \angle F$

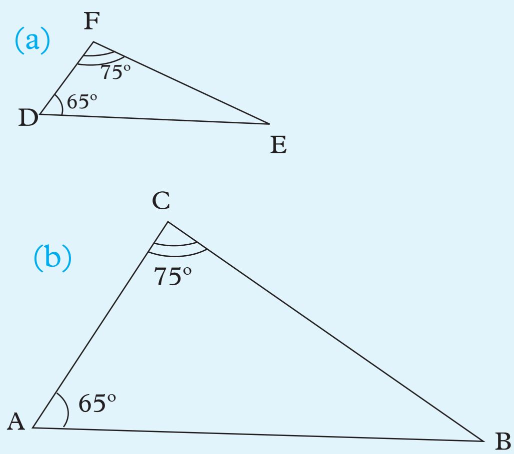


Fig. 11.4

2. Measure the remaining one angle in each triangle and compare.
3. Prove whether ΔABC is similar to ΔDEF.

Activity 11.4

1. Δs ABC and DEF shown in Fig. 11.15 are drawn accurately. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

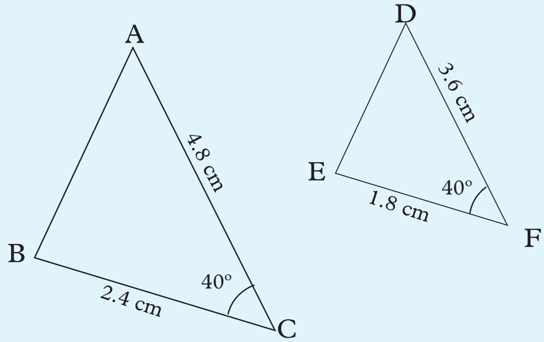


Fig. 11.5

2. Measure and record the values of the remaining side in each triangle.
3. Prove whether Δs ABC is similar to ΔDEF

Let us consider two similar triangles ΔABC and ΔDEF. (Fig 11.6)

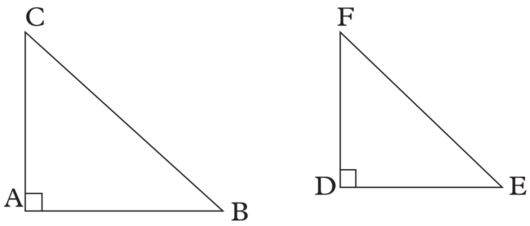


Fig. 11.6

By measurement,

1. $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ — All corresponding angles are equal

2. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ — Ratio of corresponding sides is a constant

We say that triangles ABC and DEF are similar.

Generally, two triangles are similar if:

1. If two pairs of corresponding angles are equal, then the other remaining pair of corresponding angles must also be equal.
2. If the ratios of two pairs of corresponding sides is the same and angles the between those considered sides in each triangle are equal, then the ratio will also be the same one for the remaining two pairs of corresponding sides.

Similarity is denoted by the symbol "~". So, for the above two similar triangles, we write:

ABC~ΔDEF

Mathematically, two triangles are said to be similar, if one of the following three criteria hold:

1. AAA or AA criterion: Two triangles are similar if either all the three corresponding angles are equal or any two corresponding angles are equal. AAA and AA criteria are same because if two corresponding angles of two triangles are equal, then third corresponding angle will definitely be equal.
2. SSS criterion: Two triangles are said to be similar, if all the corresponding sides are in the same proportion.
3. SAS criterion: Two triangles are similar if their two corresponding

sides are in the same proportion and the corresponding angles between these sides are equal.

Similar figures have the same shape irrespective of the size.

Note: Would the statement still be true if 'triangles' is replaced with 'polygons'? The answer is 'No': With all polygons other than triangles, the 'or' must be replaced with 'and'.

Example 11.1

In ΔKLM (Fig. 11.7), PQ is parallel to LM. Prove that ΔKPQ is similar to ΔKLM

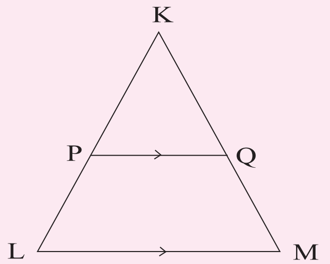


Fig. 11.7

Solution

In Fig. 11.8 we are given that PQ// LM, then

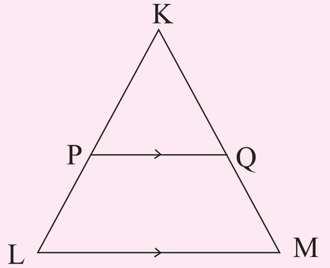


Fig. 11.8

∠KPQ = ∠KLM (corresponding angles)

∠KQP = ∠KML (corresponding angles)

∠PKQ = ∠LKM (same angle)

Then ΔKPQ is similar to ΔKLM

Hence, $\frac{KP}{KL} = \frac{KQ}{KM} = \frac{PQ}{LM}$

Exercise 11.1

1. State if, and why, the pairs of shapes in Fig. 11.9 are similar.

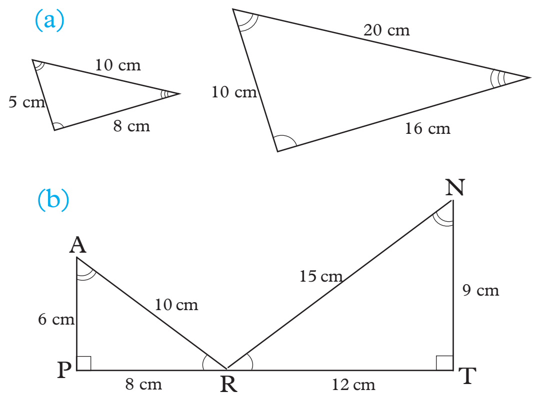


Fig. 11.9

2. Construct two triangles ABC and PQR with sides 3 by 3 by 5 cm and 12 by 12 by 20 cm respectively. Measure all the angles. Are triangles ABC and PQR similar? Are all isosceles triangles similar? Are all equilateral triangles similar?

3. The vertices of three right-angled triangles are given below:
   A(3 , 3), B(4 , 5), C(3 , 5);
   P(1 , 3), Q(1 , 5), R(2 , 4);
   X(–2 , 3), Y(1 , –1), Z(–2 , –1).
   Which two triangles are similar?

4. In Fig. 11.10 ∠QFR = 90° and ∠QER = 90°.

   Prove that $\frac{FE}{QR} = \frac{PE}{PR}$.

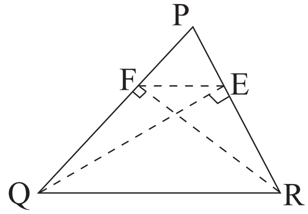


Fig. 11.10

5. In Fig. 11.11, LR//MQ. If the line joining PR is parallel to MK, prove that, $\frac{TR}{MP} = \frac{PQ}{MQ}$.

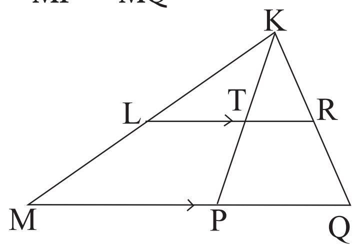


Fig. 11.11

11.1.2 Similar polygons

Activity 11.5

1. (a) Draw a rectangle ABCD measuring 3 cm by 4 cm.
   (b) Draw another rectangle EFGH measuring 4 cm by 6 cm.
   (c) What do you notice about the two rectangles? Find the ratio of the corresponding sides.
   (d) What do you notice?
2. (a) Draw a rectangle PQRS measuring 2 cm by 4 cm.
   (b) Draw another rectangle UVWX measuring 5 cm by 10 cm.
   (c) Find the ratio of corresponding sides of the rectangles.
   (d) What do you notice?

Two or more polygons are similar if the ration of the corresponding sides is constant and the corresponding angle are equal.

Example 11.2

Consider rectangle WXYZ of length 2 cm and width 3 cm and rectangle QRST of the lengths 3 cm width 5 cm. Find the ratio of their corresponding sides.

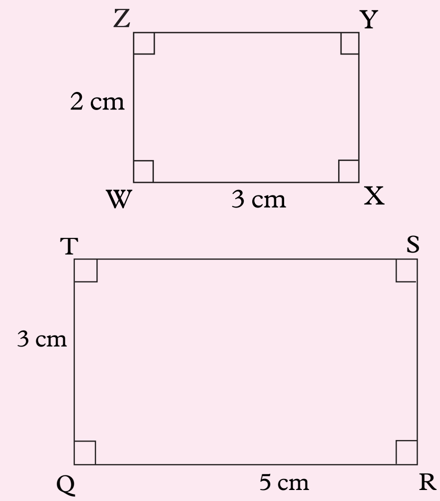


Fig. 11.12

Solution

The corresponding angles are all equal
(i.e $\angle W = \angle Q = 90^o$, $\angle X = \angle R$, $\angle Y = \angle S$ and $\angle Z = \angle T = 90^o$).
The ratios of the corresponding sides is not constant i.e. $\frac{WX}{QR} = \frac{3}{5}, \frac{XY}{SR} = \frac{2}{3}$. *Thus,* $\frac{WX}{QR} \neq \frac{XY}{SR}$.
The two rectangles are not similar.

Activity 11.6

Consider the shapes in Fig. 11.13 below.

1. Determine by measurements which shapes in the figure 11.13 are similar to (a)?

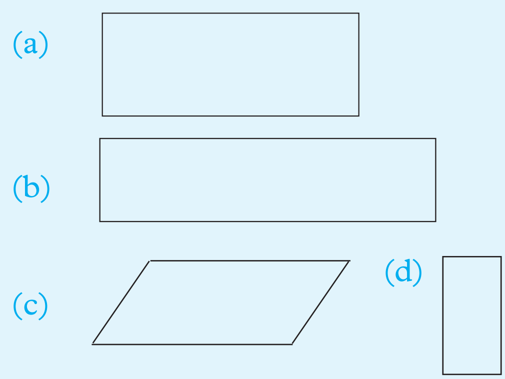


Fig. 11.13

2. Why are others not?

Example 11.3

Are the objects in Fig 11.14 (a) and (b) below similar? Justify your answer.

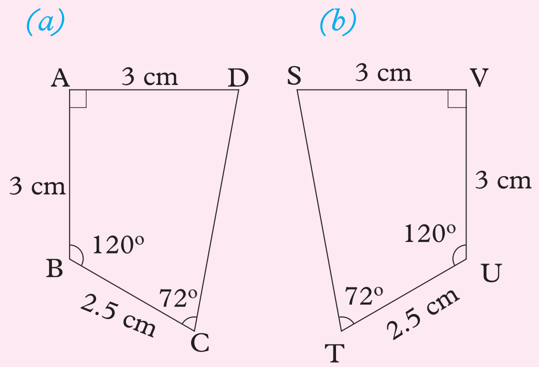


Fig 11.14

Solution

Fig 11.14 (a) and (b) are similar.

To justify if they are similar, we proof whether the corresponding angles are equal and if the ratio of the corresponding sides is equal or the same.

In Fig. 11.14 ∠ A corresponds to ∠ V = 90°

∠B corresponds to ∠U = 120°

∠C corresponds to ∠T = 72°

∠D corresponds to

$\angle S = 360^o - (90^o + 120^o + 72^o) = 78^o$

Also, $\frac{AB}{UV} = \frac{BC}{TU} = \frac{CD}{ST} = \frac{DA}{VS} = 1$*. Thus, the two objects are similar.*

Note: Any two polygons are said to be similar if the two conditions for similarity holds i.e. corresponding angles and the ratio of corresponding sides is the same or constant.

Example 11.4

Consider the shapes in Fig 11.15. Which among these shapes are similar to Fig 11.15 (a).

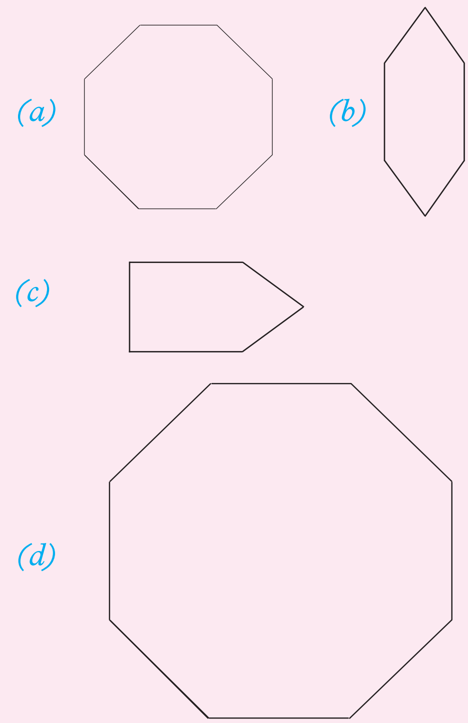


Fig. 11.15

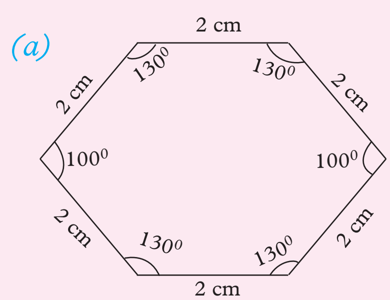
Solution

Fig 11.15 (d) is similar to Fig 11.15 (a).

The corresponding angles are equal and the ratio of the corresponding sides is constant.

Example 11.5

Determine whether the hexagons in Fig. 11.16 are similar. State your reasons.



(b)

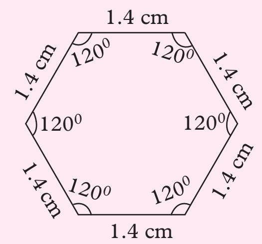


Fig. 11.16

Solution

In Fig. 11.16 (a), each side is 2 cm. In (b), each side is 1.4 cm.

∴ Ratio of corresponding sides is $\frac{1.4}{2}$ = 0.7 (a constant)

In (a), there are two angles of 100°, and four angles of 130° each. Each angle in (b) is 120°.

∴ not all corresponding angles are equal.

Although the ratio of corresponding sides is constant, not all corresponding angles are equal.

Hence, the two hexagons are not similar.

Remember that:

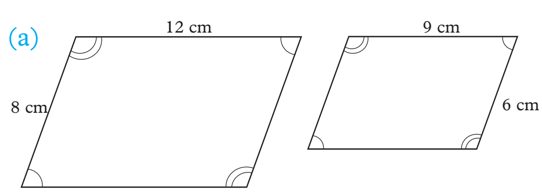
Two polygons are similar if:

1. The ratio of corresponding sides is constant
2. The corresponding angles are equal.

Exercise 11.2

1. State if, and why, the pairs of shapes in Fig. 11.17 are similar.

(a)



(b)

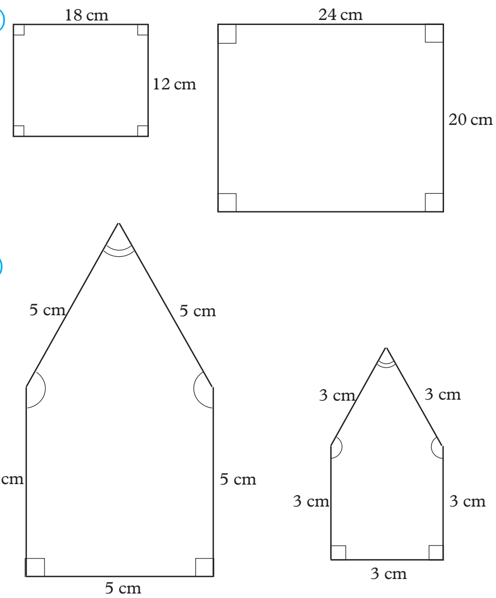


Fig. 11.17

2. Measure the length and breadth of this text book. Measure also the length and breadth of the top of your desk. Are the two shapes similar? If not, make a scale drawing of a shape that is similar to the top of your desk.
3. A photograph which measures 27 cm by 15 cm is mounted on a piece of card so as to leave a border 2 cm wide all the way round. Is the shape of the card similar to that of the photograph? Give reason for your answer.

11.1.3 Calculating lengths of sides of similar shapes using similarity and Thales theorem

Activity 11.7

Consider the rectangles PQRS and WXYZ in Figure 11.18 below.

Consider the rectangles PQRS and WXYZ in Figure 11.18 below.

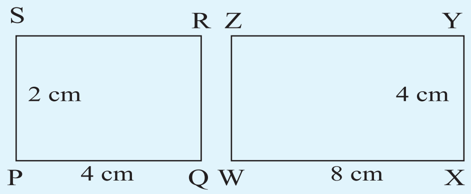


Fig. 11.18

(a) Find the values of $\frac{SP}{ZW}$ and $\frac{PQ}{WX}$

(b) What do you notice between the two results? Make a conclusion.

When two shapes e.g. triangles ABC and PQR are similar

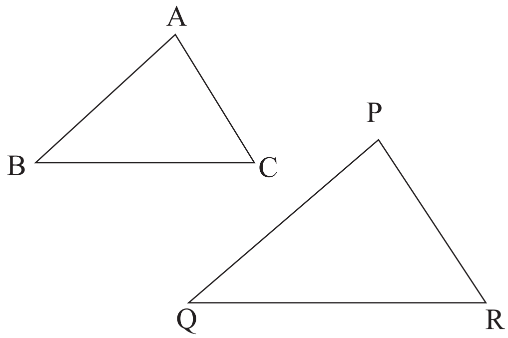


Fig. 11.19

the ratio of corresponding sides is constant, i.e,

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$$

We can use this fact to calculate the lengths of sides of the triangles.

"If a line is drawn parallel to one side of a triangle intersecting the other two sides, it divides the two sides in the same ratio.

For example, in triangle PQR below, ST is parallel to PQ.

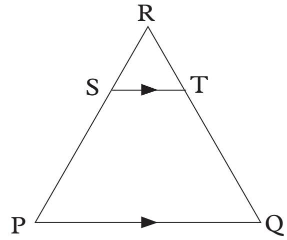


Fig. 11.20

Thales' theorem states that:

$$\frac{PS}{SR} = \frac{QT}{TR}$$

For trapezium figure 11.21 below;

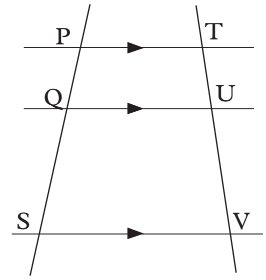


Fig. 11.21

Thales' theorem states that:

$$\frac{PQ}{QS} = \frac{TU}{UV}$$

Example 11.6

Show that triangle ABC in Fig. 11.23 is similar to triangle LMN in Fig. 11.22 below. Hence find the values of x and y.

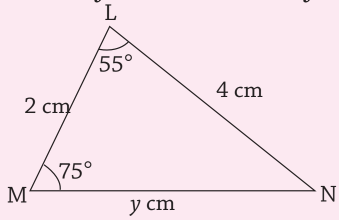


Fig. 11.22

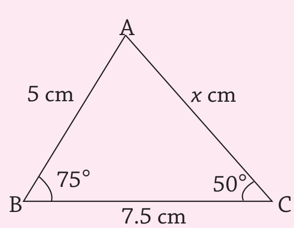


Fig. 11.23

Solution

(i) *LM corresponds to AB, LN corresponds to AC and MN corresponds to BC.*

$\therefore \frac{AB}{LM} = \frac{AC}{LN} = \frac{BC}{MN}$ = *constant ratio (L.S.F).*

$\angle LMN = \angle ABC = 75°$

$\angle MLN = \angle BAC$

$= 180^o - (75° + 50°)$

$= 55°$

$\angle LNM = 180 - (55° + 75°) = 50°$, $\angle ACB = 50°$

$\therefore \Delta LMN$ and ΔABC are similar.

Thus, $\frac{AC}{LN} = \frac{AB}{LM} \Rightarrow \frac{x}{4} = \frac{5}{2}$

$2x = 20$

$x = 10$ cm

$\frac{BC}{MN} = \frac{AB}{LM} \Rightarrow \frac{7.5}{y} = \frac{5}{2}$

$5y = 15$

$y = 3$ cm

Example 11.7

In Fig. 11.24, BE is parallel to CD. AC = 12 cm, BE = 5 cm, CD = 8 cm and AE = 6 cm. Find x and y.

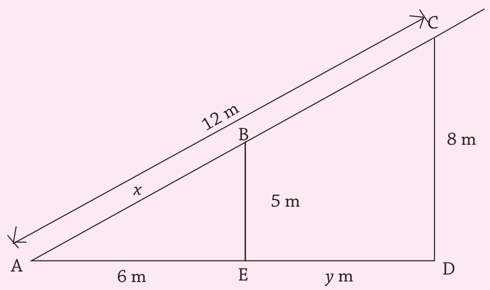


Fig. 11.24

Solution

Separating the triangles

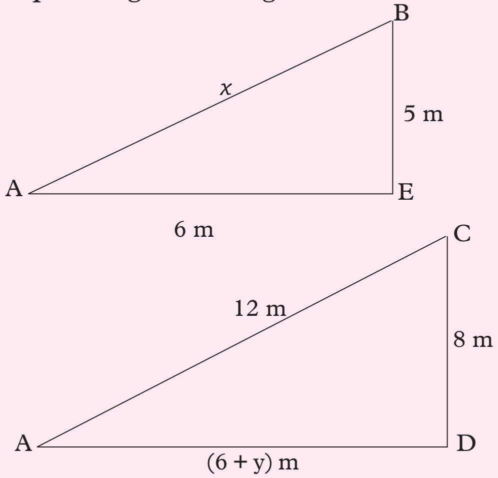


Fig. 11.25

AC corresponds to AB; AD corresponds to AE and CD corresponds to BE

$\angle AEB = \angle ADC = 90°$ and $\angle CAD = \angle BAE$

$\Rightarrow \Delta AEB$ and ΔADC are similar.

$\frac{AB}{AC} = \frac{BE}{CD} = \frac{AE}{AD}$

$\Rightarrow \frac{x}{12} = \frac{5}{8} = \frac{6}{6+y}$

$\frac{AB}{AC} = \frac{BE}{CD}$

$\therefore \frac{x}{12} = \frac{5}{8}$

$8x = 60$

$x = 7.5$ m

Also,

$\frac{AE}{AD} = \frac{BE}{CD}$

$\frac{6}{6+y} = \frac{5}{8}$

$30 + 5y = 48$

$5y = 18$

$y = 3.6$ m

Example 11.8

Fig. 11.26 and Fig. 11.31 below are similar. Find the lengths of sides marked with letters.

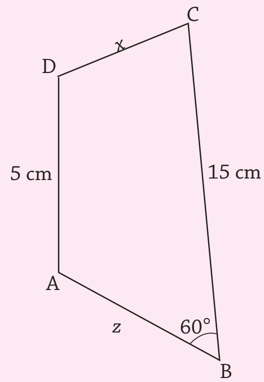


Fig. 11.26

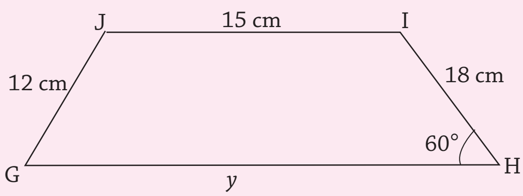


Fig. 11.27

Solution

$\frac{JI}{DA} = \frac{IH}{AB} \Rightarrow \frac{15}{5} = \frac{18}{z}$

$15z = 90$

$z = \frac{90}{15} = 6$ cm

$\frac{JI}{DA} = \frac{GH}{CB}$

$\frac{15}{5} = \frac{y}{15}$

$y = 3 \times 15$

$y = 45$ cm

$\frac{JI}{DA} = \frac{JG}{DC}$

$\frac{15}{5} = \frac{12}{x}$

$3 = \frac{12}{x} \Rightarrow 3x = 12$

$x = 4$ cm

Example 11.9

Fig. 11.28 shows two triangles ABC and PQR. Calculate the lengths BC and PQ.

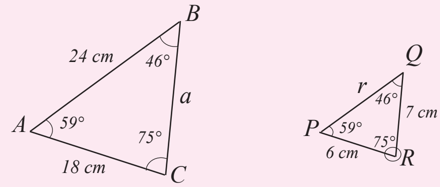


Fig. 11.28

Solution

Since the corresponding angles are equal, Δs ABC and PQR are similar.

$\therefore$ the ratio of corresponding sides is constant.

Thus, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$

i.e. $\frac{24}{r} = \frac{a}{7} = \frac{18}{6}$

$\Rightarrow \frac{a}{7} = \frac{18}{6}$ i.e. $\frac{a}{7} = 3$

$\therefore a = 21$ i.e. $BC = 21$ cm

Also, $\frac{24}{r} = \frac{18}{6}$ i.e. $\frac{24}{r} = 3$

$\Rightarrow 3r = 24$

$\therefore r = 8$ i.e. $PQ = 8$ cm

Example 11.10

In the figure below, DE is parallel to BC, find the value of x.

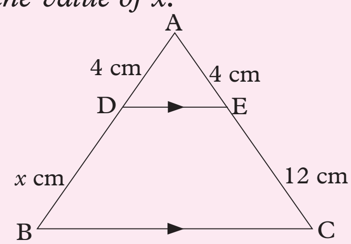


Fig. 11.29

Solution

By Thales' theorem $\frac{AD}{DB} = \frac{AE}{EC}$

$\Rightarrow 4x = 48$ and $x = 12$

Example 11.11

Find the value of y in the figure below.

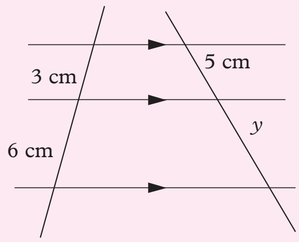


Fig. 11.30

Solution

By using Thales' theorem,

We get $\frac{3}{6} = \frac{5}{y}$.

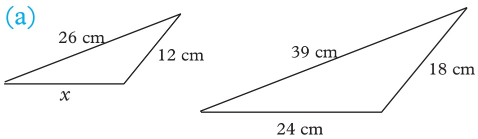
By cross-multiplying, we get $3y = 30$

We get $y = 10$.

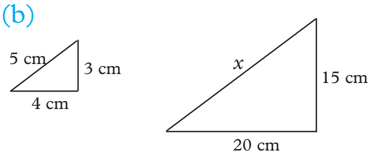
Exercise 11.3

1. The triangles in each pair in Fig. 11.31 are similar. Find x.

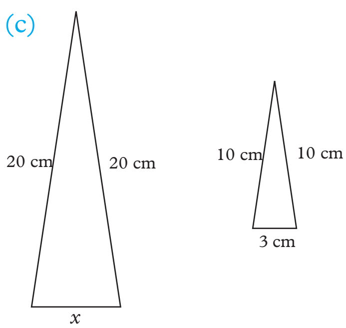
(a)



(b)



(c)



(d)

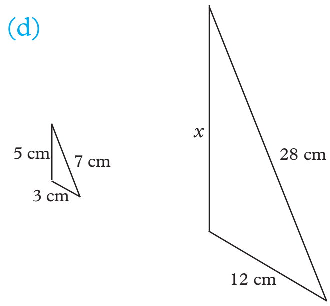


Fig. 11.31

2. Show that the two triangles in Fig. 11.32 are similar. Hence calculate AC and PQ.

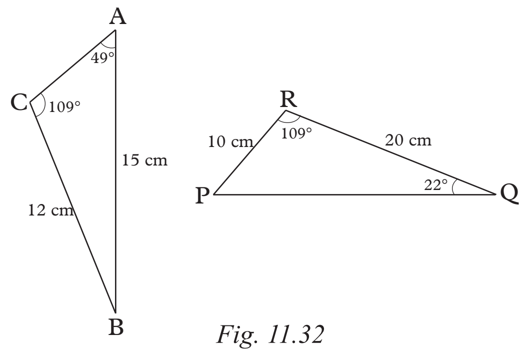
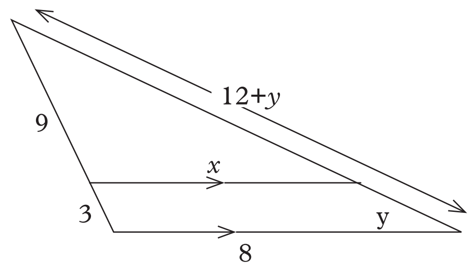


Fig. 11.32

3. Find the values of x and y in Figure 11.33 (a) and (b).

(a)



(b)

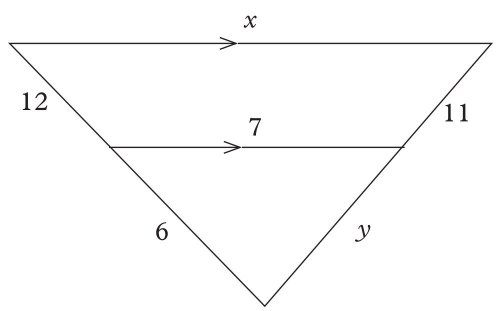


Fig. 11.33

4. Which two triangles in Fig. 11.34 are similar? State the reason. If AB = 6 cm, BC = 4 cm and DE = 9 cm, calculate BD.

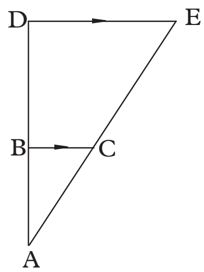


Fig. 11.34

5. Find the value of x from the figures 11.35 and 11.36 below:

(a)

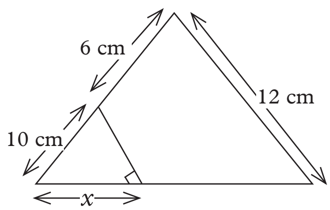


Fig. 11.35

(b)

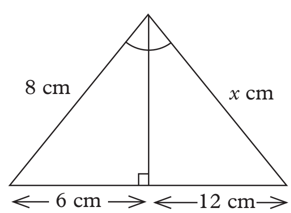


Fig. 11.36

6. In Fig. 11.37, identify two similar triangles. Use the similar triangles to calculate x and y.

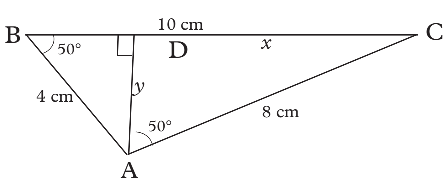


Fig. 11.37

7. A student used similar triangles to find the distance across a river. To construct the triangles she made the measurements shown in Fig. 11.38. Find the distance across the river.

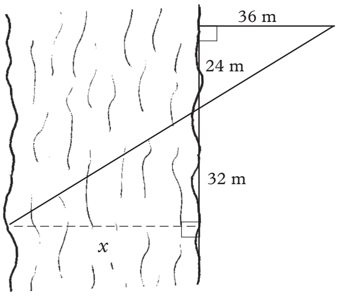


Fig. 11.38

11.1.4: Definition and properties of similar solids

Activity 11.8

Consider Fig. 11.39 (a) and 11.2 (b).

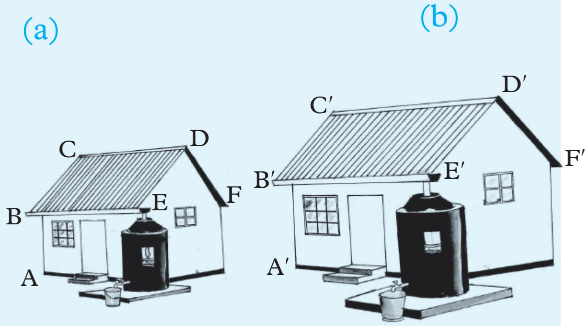


Fig. 11.39

1. Measure the lengths of the corresponding lines BC and B′C′, CD and C′D′, DE and D′E′, DF and D′F′, etc.
2. Copy and complete the following.

$\frac{B'C'}{BC} = \qquad \frac{C'D'}{CD} =$

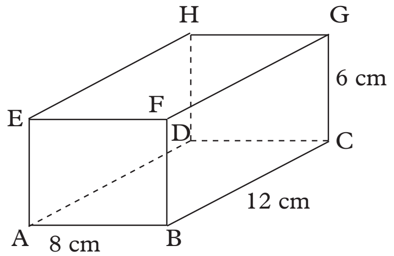
$\frac{D'C'}{DE} = \qquad \frac{D'F'}{DF} =$

3. What do you notice about the ratio of corresponding sides?
4. Measure the corresponding angles ∠A and ∠A′, ∠B and ∠B′, ∠C and ∠C′, etc.

   What do you notice?

Consider a small cuboid measuring 8 cm × 12 cm × 6 cm and big cuboid measuring 12 cm × 18 cm × 9 cm shown in Fig 11.40.

(a)



(b)

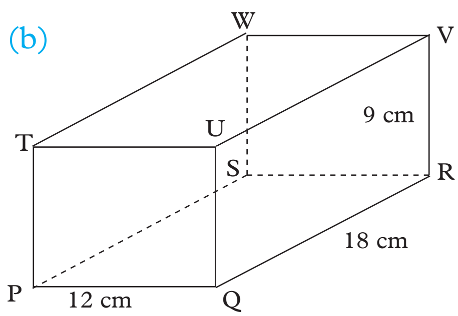


Fig 11.40

Lets compare the ratios of corresponding sides

$\frac{PQ}{AB} = \frac{12\text{ cm}}{8\text{ cm}} = 1.5$, $\frac{QR}{BC} = \frac{18\text{ cm}}{12\text{ cm}} = 1.5$,

$\frac{RV}{CG} = \frac{9\text{ cm}}{6\text{ cm}} = 1.5\text{ cm}$,

We notice that the ratio of corresponding sides is a constant (same).

Lets compare the corresponding angles.

∠ABC = ∠PQR, ∠ABF = ∠PQV,

∠BCG = ∠QRV

We notice that the all corresponding angles are equal.

We say that cuboid ABCDEFGH is similar to cuboid PQRSTUVW.

Note that even naming similar figures we match the corresponding sides e.g. AB corresponds to PQ, CD coresponds to RS and so on.

Two solids are similar if:

1. The ratio of the lengths of their corresponding sides is constant.
2. The corresponding angles are equal.

Example 11.12

Determine whether the following pairs objects are similar or not.

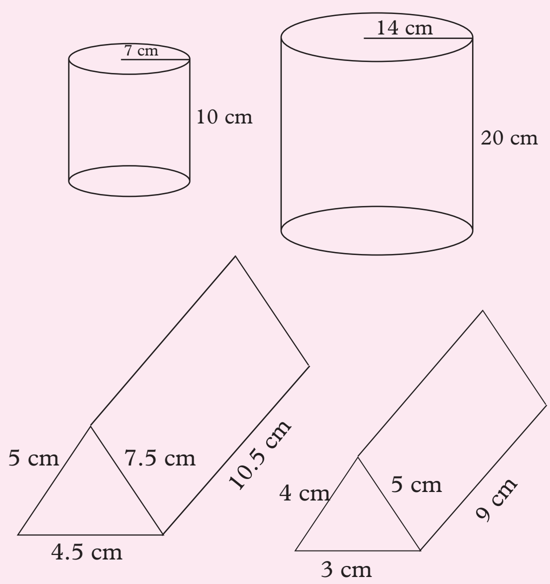


Fig. 11.41

Solution

(a) $\frac{14 \text{ cm}}{7 \text{ cm}} = 2, \frac{20 \text{ cm}}{10 \text{ cm}} = 2$

The two cylinders are similar

(b) $\frac{10.5 \text{ cm}}{9 \text{ cm}} = 1.167, \frac{7.5 \text{ cm}}{5 \text{ cm}} = 1.5$

$\frac{5 \text{ cm}}{4 \text{ cm}} = 1.25, \frac{4.5 \text{ cm}}{3 \text{ cm}} = 1.5$

Not all ratios of corresponding sides are equal i.e the ratio of corresponding sides is not a constant.

Hence, the two prisms are not similar.

Example 11.13

A jewel box, of length 30 cm, is similar to a matchbox. If the matchbox is 5 cm long, 3.5 cm wide and 1.5 cm high, find the breadth and height of the jewel box in Fig. 11.42.

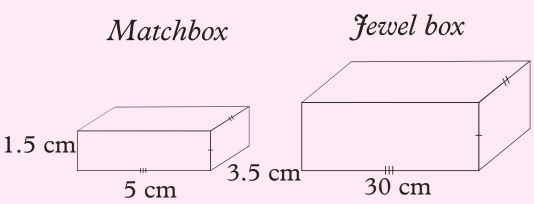


Fig. 11.42

Solution

The ratio of the lengths of the jewel box and the matchbox is

$$\frac{\text{Length of jewel box}}{\text{Length of match box}} = \frac{30 \text{ cm}}{5 \text{ cm}} = 6$$

This means that each edge of the jewel box is 6 times the length of the corresponding edge of the matchbox.

Width of jewel box $= 6 \times 3.5 \text{ cm} = 21 \text{ cm}$, *and height of jewel box* $= 6 \times 1.5 \text{ cm} = 9 \text{ cm}$.

Exercise 11.4

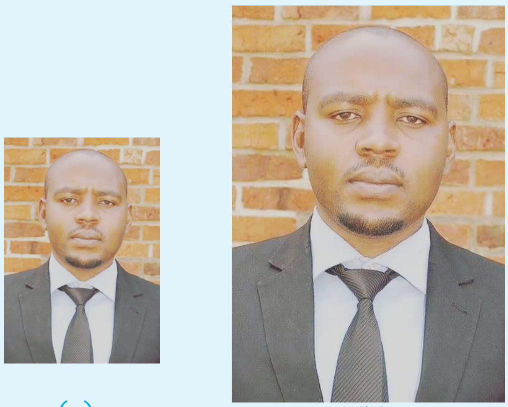
1. A scale model of a double-decker bus is 7.0 cm high and 15.4 cm long. If the bus is 4.2 m high, how long is it?
2. A cuboid has a height of 15 cm. It is similar to another cuboid which is 9 cm long, 5 cm wide and 10 cm high. Calculate the area of the base of the larger cuboid.
3. A water tank is in the shape of a cylinder radius 2 m and height 3 m. A similar tank has a radius of 1.5 m. Calculate the height of the smaller tank.
4. Write down the dimensions of any two cubes. Are the two cubes similar? Are all cubes similar? Are all cuboids similar?
5. A designer has two models of a particular car. The first model is 15 cm long, 7.5 cm wide and 5 cm high. The second model is 3.75 cm long, 1.70 cm wide and 1.25 cm high. He says that the two are 'accurate scale models'. Explain whether or not his claim could be true.

11.2 Enlargement

11.2.1 Definition of enlargement

Activity 11.9

Observe the two pictures in Fig. 11.43.



(a) (b)

Fig 11.43

1. Compare the shapes and looks of the two pictures. What do you notice?
2. By measuring, determine how many times picture (b) is bigger than picture (a)
3. What is the name of the transformation that transforms Fig. 11.43 (a) to (b)?
4. Identify the requirements for the transformation to be performed?

Consider the Triangles in Fig.11.44.

(a)

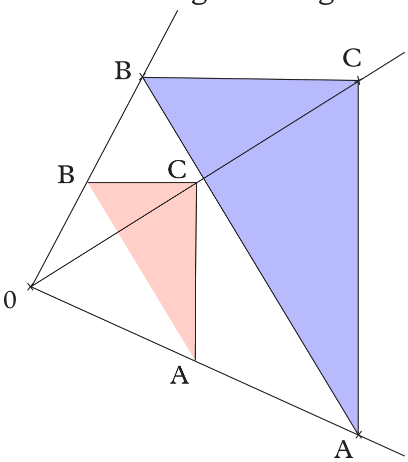


Fig 11.44

The two triangles have exactly the same shape. However, triangle A′B′C′ is bigger than triangle ABC. We say that the triangle ABC has been enlarged to triangle A′B′C′. Enlargement is the transformation that changes the size of an object but preserves its shape i.e angles are preserved.

Lines AA′, BB′ and CC′ produced meet at a common point 0. The point is called the centre of enlargement. Triangle ABC and A′B′C′ are similar.

By measuring, determine the value of $\frac{OA'}{OA}$. This is called the scale factor of enlargement.

The scale factor of enlargement is defined as the ratio.

$$\frac{\textit{Linear size of image}}{\textit{Corresponding linear size of object}}, \text{ or}$$

$$\frac{\textit{Distance of an image point from the centre of enlargement}}{\textit{Distance of a corresponding object point from the centre of enlargement}}$$

Note:

This particular scale factor is often called the **linear scale factor** because it is the ratio of the lengths of two **line** segments.

Thus, the scale factor of enlargement (k) tells us:

(a) How big or small the image is compared to the object.

(b) How far and in which direction is the image in respect to the object.

Therefore, for an enlargement to be performed, the centre and scale factor of enlargement must be known.

11.2.2 Constructing objects and images under enlargement

11.2.2.1 Positive scale factor > 1

Activity 11.10

1. Draw any triangle ABC.
2. Taking a point O outside the triangle as the centre of enlargement, and with a scale factor 3, construct the image triangle A′B′C′.

The following is the procedure of constructing the image of triangle XYZ under enlargement scale factor.

(a) Draw triangle XYZ and choose a point O outside the triangle.

(b) Draw construction lines OX, OY and OZ, and produce them (Fig. 11.45).

(c) Measure OX, OY and OZ. Calculate the corresponding lengths OX′, OY′ and OZ′; and mark off the image points. From the definition of scale factor, it follows that image distance = k × object distance, where k is the scale factor. Thus, OX′ = 3OX, OY′ = 3OY and OZ′ = 3OZ.

(d) Join points X′, Y′, Z′ to obtain the image triangle.

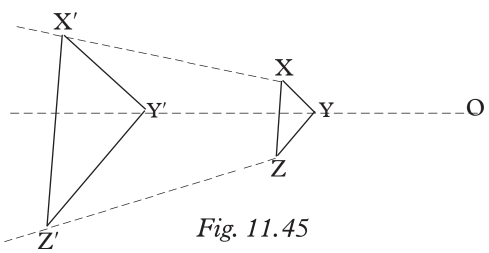


Fig. 11.45

Example 11.14

Enlarge triangle PQR with a scale factor of +3 and C as the centre of enlargement.

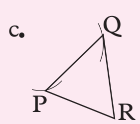


Fig. 11.46

Solution

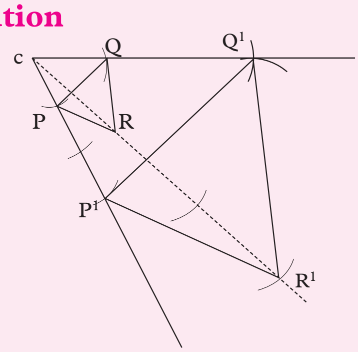
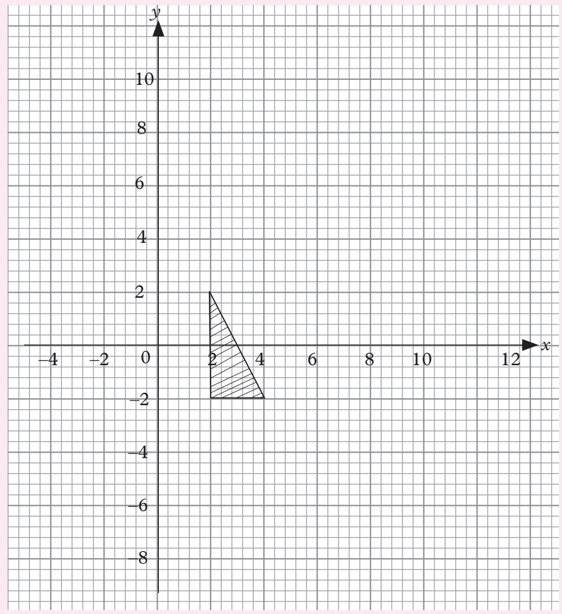


Fig. 11.47

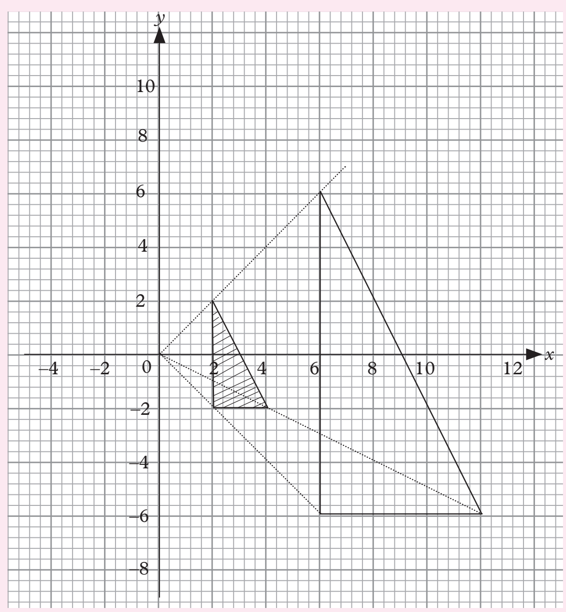
Example 11.15

Enlarge the triangle ABC given in the grid (Graph 11.1) by a scale factor 2 with centre of enlargement at (0,0).



Graph 11.1

Solution



Graph 11.2

Exercise 11.5

1. Copy each of the shapes in Fig. 11.48. Using the centre of enlargement indicated as X, enlarge each of the shapes with a scale factor of 3.

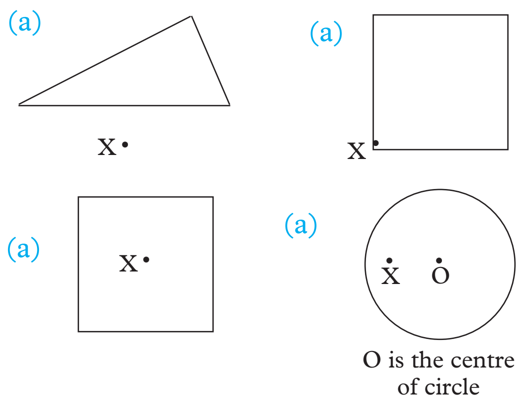


Fig. 11.48

2. In Fig. 11.49, PQRS is an enlargement of PTUV.
   - (a) Which point is the centre of enlargement?
   - (b) By measurement, find the scale factor of enlargement.

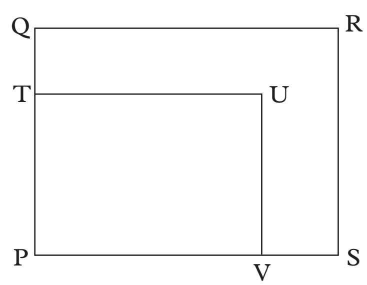


Fig. 11.49

3. Find the image of triangle ABC (Fig. 11.50) below under an enlargement scale factor 2 centre O.

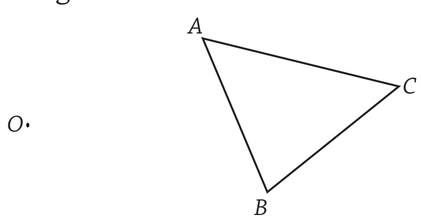


Fig. 11.50

11.2.2.2 Fractional Scale factor

Activity 11.10

Construct the enlargement of triangle PQR shown in Fig. 11.51 with O as the centre of enlargement and scale factor $\frac{1}{3}$.

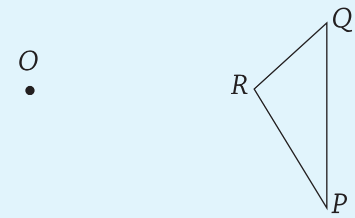


Fig. 11.51

Example 11.16

Construct any triangle ABC. Choose a point O outside, and a bit far from the triangle. With O as the centre of enlargement, and with a scale factor $\frac{1}{2}$, construct the image triangle $A'B'C'$.

Solution

The procedure is the same as that of Example 11.14.

The only difference is that we multiply the object distance by $\frac{1}{2}$ (a fraction) so as to get the image distance, i.e.

i.e. Image distance $= \frac{1}{2} \times$ Object distance.
The construction is shown in Fig. 11.51.

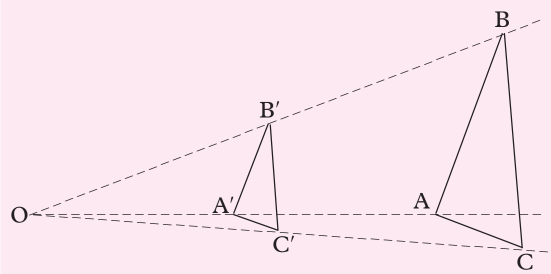


Fig. 11.52

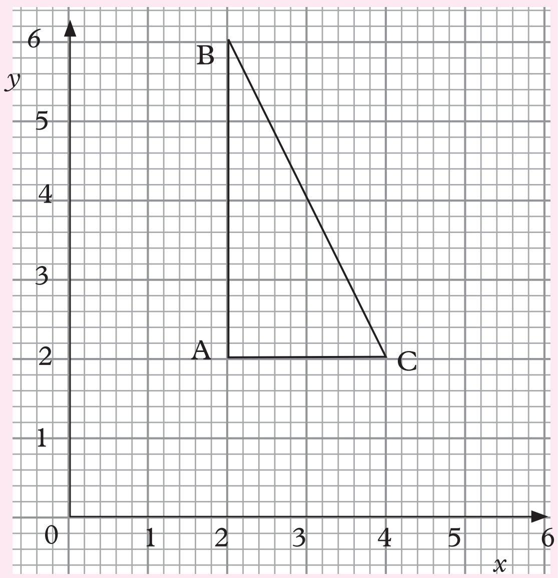
We notice that:

Under enlargement with a **scale factor that is a proper fraction***:*

1. The object and the image are on the same side of the centre of enlargement.
2. The image is smaller than the object and lies between the object and the centre of enlargement.
3. Any line on the image is parallel to the corresponding line on the object.

Example 11. 17

Enlarge triangle ABC with a scale factor $\frac{1}{2}$, centered about the origin.



Graph 11.3

Solution

The scale factor is $\frac{1}{2}$, so:

$OA' = \frac{1}{2}OA$

$OB' = \frac{1}{2}OB$

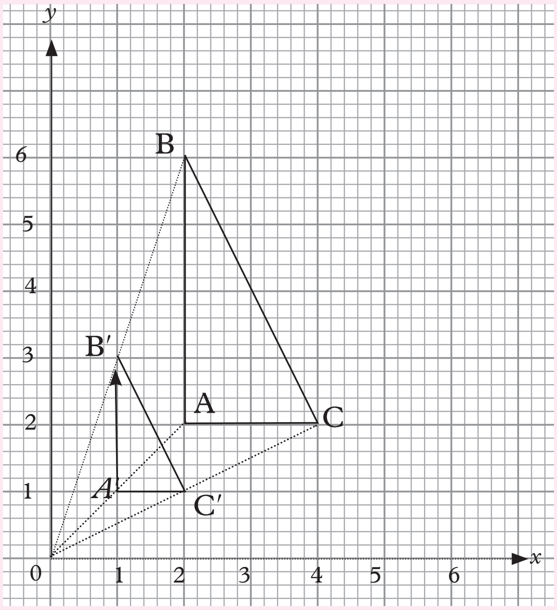
$OC' = \frac{1}{2}OC$

Since the centre is the origin, we can in this case multiply each coordinate by $\frac{1}{2}$ to get the answers.

A = (2, 2), so A′ will be (1, 1).

B = (2, 6), so B′ will be (1, 3).

C = (4, 2), so C′ will be (2, 1).



Graph 11.4

Exercise 11.6

1. Copy Fig. 11.53 and locate the image of the flag under enlargement with centre O and scale factor (a) $\frac{1}{4}$ (b) 2.

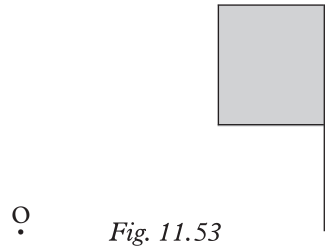


Fig. 11.53

2. A′(1 , 4), B′(–1 , 2), C′(2 , 2) are the

vertices of the image of a triangle with vertices A(1 , 2), B(–5 , –4), C(4, –4) under a certain transformation.

Draw the two triangles on squared paper and fully describe the transformation.

3. In Fig. 11.54, P′Q′ is an enlargement of PQ with centre O.

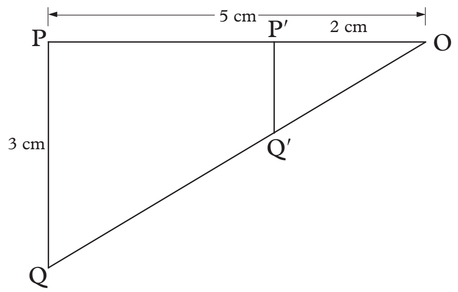


Fig. 11.54

(a) Find the scale factor of enlargement.
(b) Calculate the length of P′Q′.

4. Under enlargement of a polygon, what happens to the following if the scale factor is: (i) 3? (ii) $\frac{1}{3}$?
(a) The lengths of corresponding sides
(b) Corresponding angles
(c) Shape
(d) Direction of lines

11.2.2.3 Negative scale factor

Activity 11.12

In this activity, you will locate that ΔA′B′C′ as the image of ΔABC in Fig. 11.55 under enlargement centre O, L.S.F –3, locate the image.

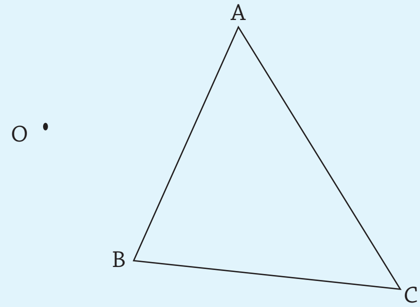


Fig. 11.55

Procedure

(i) Join each vertex of ΔABC (object points) to centre O with straight lines.
(ii) Prolong each of these lines in the opposite side from centre O.
(iii) Measure the distance of each object point to centre O.
(iv) Multiply each of the object distance by 3 to get the corresponding image distance.
(v) Using the image distances obtained in (iv) above, locate the image point of each object point on the opposite side from O on the prolonged lines.
(vi) Join the image points to form the required image triangle. Fig. 11.56 shows the required image. (ΔA′B′C′).

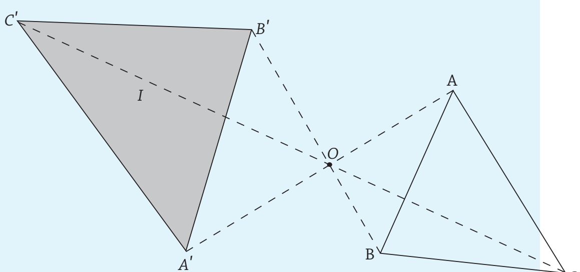


Fig. 11.56

Consider Fig. 11.57, in which the centre of enlargement is O and both images of flag ABCD are 1.5 times as large as the object.

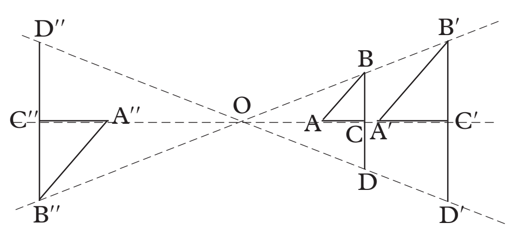


Fig. 11.57

A′B′C′D′ is the image of ABCD under enlargement with centre O and scale factor 1.5.

A″B″C″D″ is also an image of ABCD under enlargement with centre O. How is it different from A′B′C′D′?

OA = 1.2 cm and OA″ = 1.8 cm. OA and OA″ are on opposite sides of O. If we mark A″OA as a number line with O as zero and A as 1.2, then A″ is –1.8 (Fig. 11.58).

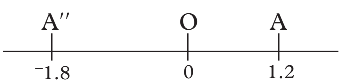


Fig. 11.58

We say that the scale factor is ⁻1.5 because –1.5 × 1.2 = –1.8.

We write OA″ = –1.5 OA.

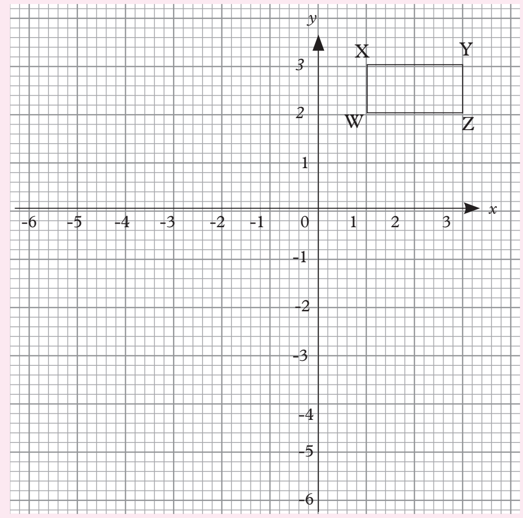
We notice that:

Under enlargement with a **negative scale factor**,

1. the object and the image are on **opposite sides** of the centre of enlargement,
2. the image is larger or smaller than the object depending on whether the scale factor is greater than 1 and negative or a negative proper fraction,
3. any line on the image is **parallel** to the corresponding line on the object, but the image is **inverted** relative to the object.

Example 11.18

Enlarge the rectangle WXYZ (Fig 11.5) using a scale factor of - 2, centered about the origin.

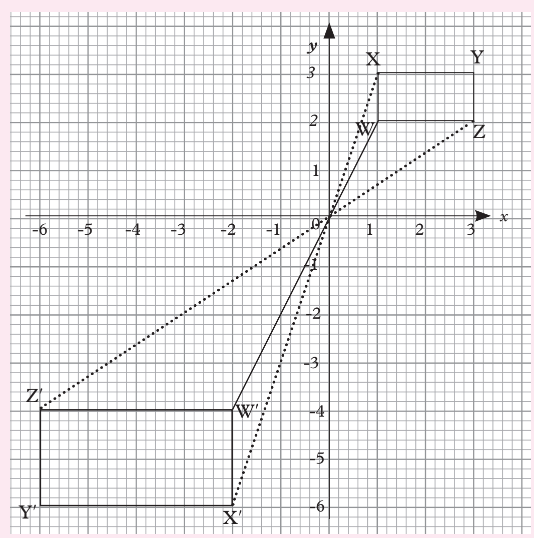


Graph 11.5

Solution

The scale factor is -2, so multiply all the coordinates by -2. So OW′ is 2OW. This time we extend the line WO beyond O, before plotting W′.

In a similar way, we extend XO, YO and ZO and plot X′, Y′ and Z′. Can you see that the image has been turned upside down?



Graph 11.6

Exercise 11.7

1. Make copies of the shapes in Fig. 11.59. Using the points marked X as centres, enlarge the shapes with scale factor

   (a) -2 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$

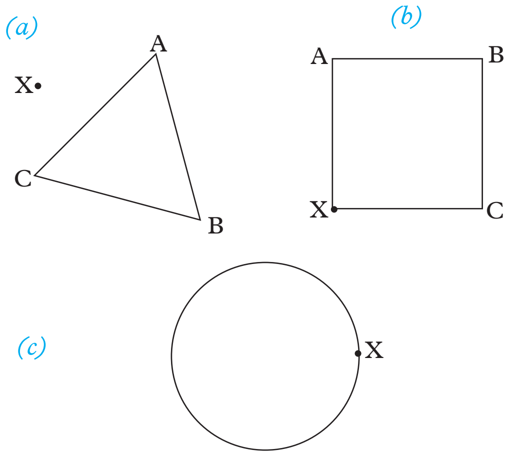


Fig. 11.59

2. Use scale factor of –4 and centre C to enlarge ΔPQR in Fig 11.60 below.

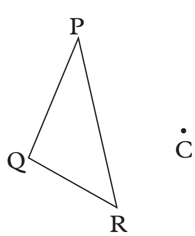


Fig. 11.60

11.2.3 Locating the centre of enlargement and finding the scale factor

Activity 11.13

1. Fig. 11.61 show triangle ABC and its image triangle A′B′C′ under the enlargement.

   (a) Locate by construction the centre of enlargement

   (b) Determine the scale of enlargement.

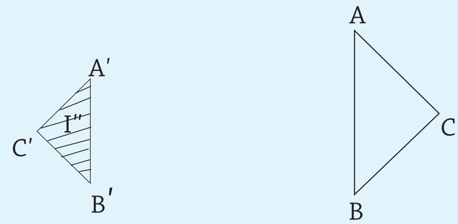


Fig. 11.61

Example 11.19

Fig. 11.62 shows a quadrilateral and its image under a certain enlargement.

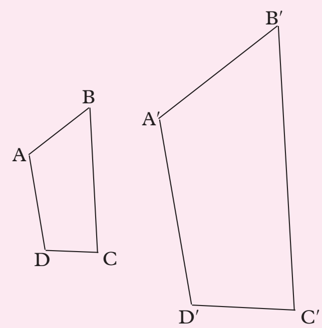


Fig. 11.62

(a) Locate the centre of enlargement.

(b) Find the scale factor of enlargement.

(c) Given that a line measures 5 cm, find the length of its image under the same enlargement.

Solution

(a) To locate the centre of enlargement O, join A to A′, B to B′, C to C′ and D to D′ and produce the lines (Fig. 11.61). The point where the lines meet is the centre of enlargement.

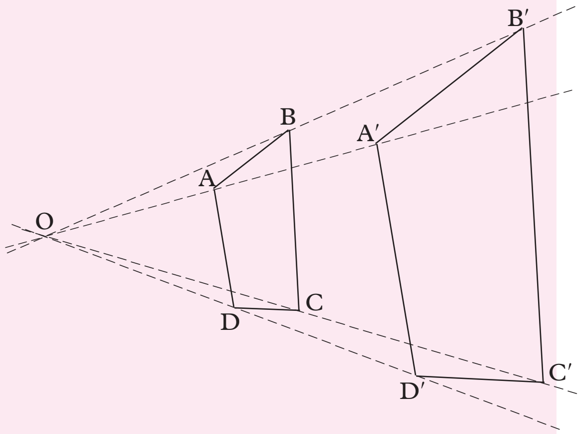


Fig. 11.63

(b) *Measure the distances OA, OA′; OB, OB′; etc. and calculate the ratios*

$\frac{OA'}{OA}, \frac{OB'}{OB}$, *etc.*

The scale factor of enlargement is 2.

(c) *Length of image*

= *length of object* × *scale factor*

= *5 × 2 = 10 cm.*

Note:

1. Given an object and its image, it is sufficient to construct only two lines in order to obtain the centre of enlargement.
2. An enlargement is fully described if the **centre and scale factor** of enlargement are given.

Exercise 11.8

1. In Fig. 11.64, ΔOA′B′ is the enlargement of ΔOAB.
   (a) Which point is the centre of enlargement?
   (b) Find the scale factor of enlargement.
   (c) Calculate the lengths A′B′ and AA′.

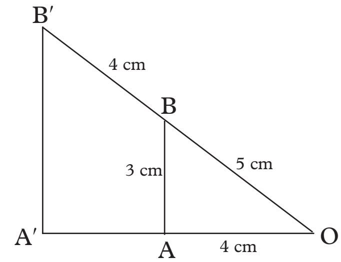


Fig. 11.64

2. In Fig. 11.65, rectangle PQRS is an enlargement of rectangle ABCD with centre O.

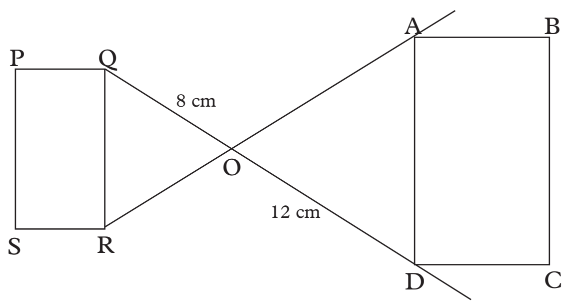


Fig. 11.65

   (a) Find the scale factor of enlargement.
   (b) Which point is the image of point (i) A; (ii) B; (iii) C?
   (c) Find the length of the diagonal of rectangle PQRS given that the length of the diagonal of rectangle ABCD is 15 cm.

3. Fig. 11.66 shows a triangle XYZ and image triangle X′Y′Z′. Copy the two triangles and by construction find:
   (a) centre of enlargement
   (b) scale factor of enlargement

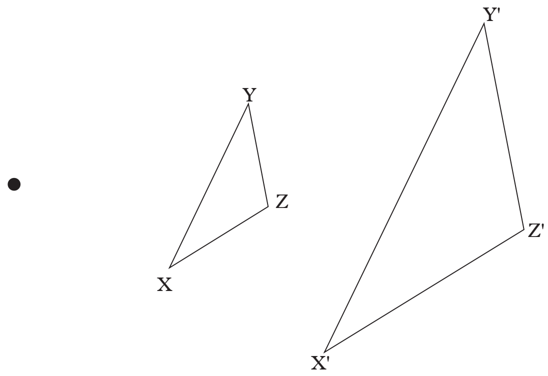


Fig. 11.66

4. Fig. 11.67 shows triangle ABC and its image triangle ABC under an enlargement.

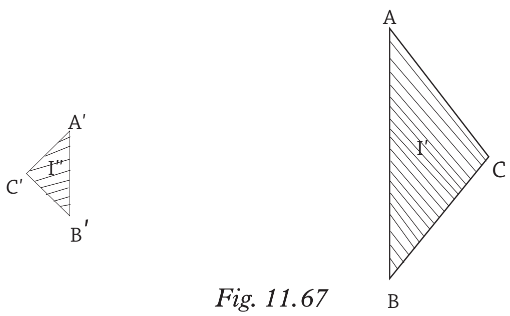


Fig. 11.67

Copy the two triangles and by construction find:
(a) centre of enlargement
(b) scale factor of enlargement.

5. In a scale model of a building, a door which is actually 2 m high is represented as having a height of 2 cm.
(a) What is the scale of this model?
(b) Calculate the actual length of a wall which is represented as being 5.2 cm long.

6. When fully inflated, two balls have radii of 10.5 cm and 14 cm respectively. They are deflated such that their diameters decrease in the same ratio. Calculate the diameter of the smaller ball when the radius of the larger ball is 10 cm.

7. A tree is 6 m high. In photographing it, a camera forms an inverted image 1.5 cm high on the film. The film is then processed and printed to form a picture in which the tree is 6 cm high. Calculate the scale factors for the two separate stages.

8. Δ A′B′C′ is the image of Δ ABC (Fig. 11.68) after an enlargement.

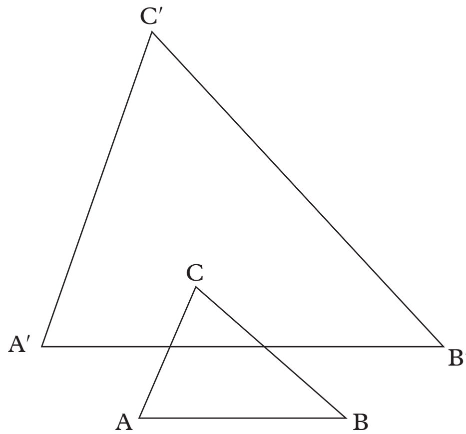


Fig. 11.68

(a) Find by construction the centre of enlargement.
(b) Find the scale factor of the enlargement.
(c) Given that a line segment measures 8.5 cm, find the length of its image under this enlargement.

11.2.4 Properties of enlargement

Activity 11.14

1. Use reference materials, internet and other resources to research on the properties of enlargement.
2. State and explain the properties.
3. Where do you think the properties apply in real life?

Having learnt how to enalarge figures using any scale factor, and to determine the centre of enlargement, confirm the following properties of enlargement as a transforming the activities we did.

1. An object point, its image and the centre of enlargement are collinear.
2. For any point A on an object, OA = kOA , where k is the scale factor.
   (a) If $k = 1$, the object id mapped onto itself.
   (b) If $k > 0$, the object and its image lie on the same side of the centre of enlargement.
   (c) If $k < 0$, the object and its image are on opposite sides of the centre of enlargement.
3. The centre of enlargement is the only point that remains fixed irrespective of the scale factor.
4. Both object and image are similar
5. If the linear scale factor of enlargement is k, the area scale factor is k^2. Where the enlargement is of a solid, the volume scale factor is k^3.

11.2.5 Enlargement in the Cartesian plane

Activity 11.15

1. On a squared paper, draw a quadrilateral with vertices A(0, 3), B(2, 3), C(3, 1) and D(3, –2).

2. Copy and complete table 11.1 for the coordinates of A′B′C′D′ and P′, the images of points A, B, C, D and a general point P(a, b) under enlargement with centre the origin and the given scale factors.

Scale factor	A′	B′	C′	D′	P′
2	(0, 6)				($2a$, $2b$)
				(1.5, –1)	
–2			(–6,–2)		
–		(–1,–1.5)			

Table 11.1

Note:

An enlargement with centre (0, 0) and scale factor k maps a point P(a, b) onto P′(ka , kb).

Activity 11.16

1. Repeat Activity 11.13, but use (2, 1) as the centre of enlargement.
2. Copy and complete Table 11.2 for the images of A, B, C and D.

Scale factor	A′	B′	C′	D′
2		(–2, 5)		
$\frac{1}{3}$				
–2				
$\frac{1}{3}$				

Table 11.2

3. What is the image of a point P(a, b) under enlargement with centre (2, 1) and each of the given scale factors?

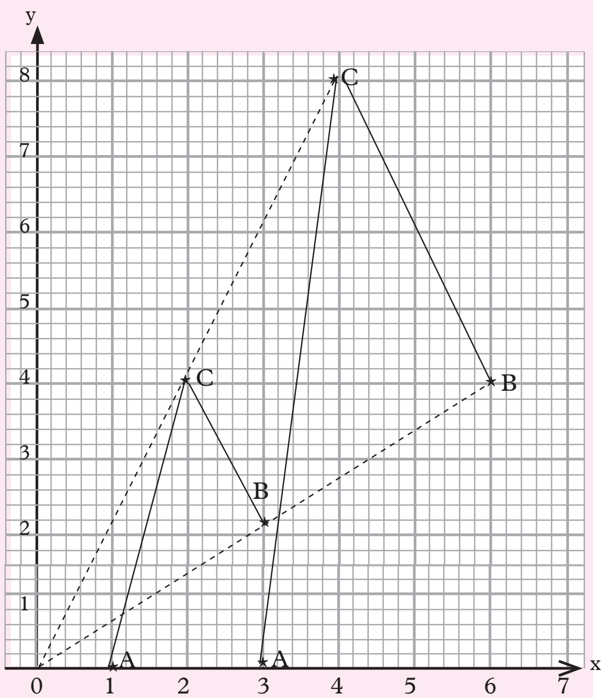
Note:
It is **not possible** to generalise for P (a , b) when the centre is **not the origin**. When the centre of enlargement is not the origin, we **must** carry out complete construction

Example 11.20

In a cartesian plane plot traingle ABC whose coordinates are A(1,0), B(3,2) and (2, 4). On the same cartesian plane locate and draw triangle A'B'C' the image of triangle ABC under enlargement scale factor 3 and centre origin.

Solution

Using the steps we learnt earlier and also fact that an enlargment centre origin scale factor K maps point P(a, b) tp P'(ka, kb) we obtain ΔA'B'C' as shown in Fig. 11.7.



Graph 11.7

Example 11.21

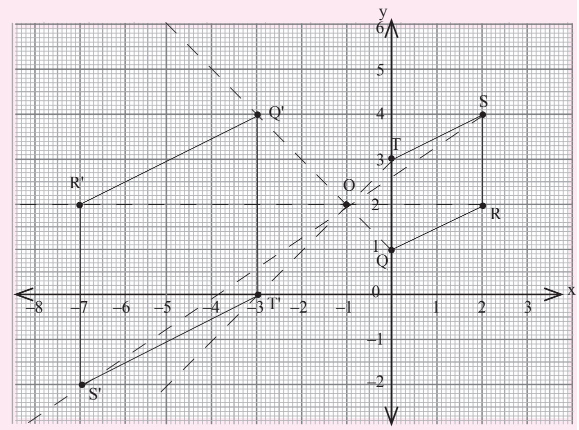
The coordinates of a quadrilateral are Q(0, 1), R(2, 2) S(2, 4) and T(0, 3).

(a) *plot the quadrilateral QRST on cartesian plane. Identify the quadrilateral by name?*

(b) *By construction, locate and draw the image of quadrilateral QRST on the cartesian plane under enlargement scale factor -2 centre (-2). Name the image as Q'R'S'T'.*

(c) *Write down the co-ordinates of vertices of the image.*

Solution



Graph 11.8

Vertices of the image Q'(–3,4), R'(–7,2), S'(–7,–2) and T'(–3,0).

Exercise 11.9

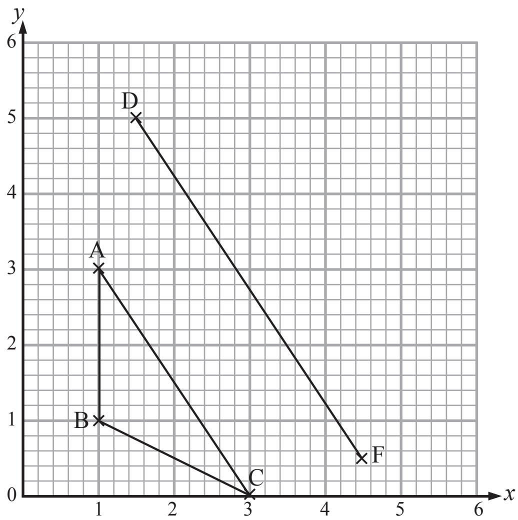
1. Points A(–2, –1), B(1,–1), C(1, –4) and D(–2, –4) are vertices of a square. Without drawing, state the coordinates of the vertices of the image square under enlargement with centre origin and scale factor:
   (a) –4 (b) –2 (c) –1 (d) $-\frac{1}{4}$
2. Triangle ABC with vertices A (9, 4), B (9, 1) and C (15, 1) is mapped onto Δ A'B'C' vertices A' (1, 2), B' (1, 1) and C' (3, 1) by an enlargement. Find:
   (a) the scale factor of the enlargement.
   (b) the co–ordinates of the centre of enlargement.

3. Triangle ABC has vertices A(1, 2), B (3, 2), C (3, 4). Find the co–ordinates of the image Δ A′B′C′ after an enlargement, centre (0, 0) scale factor –3.

4. The points A (1, 1), B (3, 2), C (3, 4) and D (1,3) are vertices of a quadrilateral.
   (a) Draw the quadilateral on the Cartesian plane.
   (b) Taking O(4, 0) as the centre of enlargement, find the image when the linear scale factor is:
   (i) 1.5 **(ii)** –2.4

5. The vertices of an object and its image after an enlargement are A (–1, 2), B (1, 4), C (2, 2) and A′ (–1, –2), B′(5, 4), C′(8, –2) respectively.
   Draw these shapes on squared paper. Hence, find the centre and scale factor of enlargement.

6. On squared paper, copy points A, B, C, D and F as they are in graph 11.9.
   Given that ΔDEF is an enlargement of ΔABC, find the coordinates of E.
   What is the centre of enlargement?



Graph 11.9

7. Points A(4, 0), B (0, 3) and C (4, 3) are the vertices of a triangle. Draw the triangle on a squared paper. Copy and complete Table 11.3 for the co–ordinates of A′, B′ and C′; the images of A, B and C under enlargement with centre (0, 0).

Scale factor	A′	B′	C′
3			
1.5			
⁻1			
$\frac{1}{2}$			

Table 11.3

8. The vertices of Δ ABC are A (3, 2), B (1, 4) and C (4, 4). Find the image of Δ ABC under enlargement, centre (0,0) and scale factor:
   (i) –2 (ii) $\frac{1}{2}$.

9. The vertices of figure ABCD are A(1, 1), B(2, 4), C(1, 5) and D(5, 4). Draw the figure and its image A′ B′ C′ D′ under enlargement scale factor +2 centre (–2, 3).

10. The vertices of ΔABC are A(5, 2), B (7, 2) and C (6,1). Draw the triangle and its image ΔA′B′C′ under enlargement scale factor –3 centre (4, 1).

11. The vertices of a triangle are X(1, 2), Y(3, 1) and Z(3, 3) and the vertices of its image are X′(–4, 4), Y′(–8, 6) and Z′(–8, 2).
    (a) Find the centre of enlargement and the scale factor.
    (b) Using the centre of enlargement in (a) above, locate the image X″Y″Z″ of XYZ using the scale factor of 2.5.

11.2.6 Area scale factor

Activity 11.17

1. Draw a rectangle. Choose a point O anywhere outside the rectangle.
2. With O as the centre, enlarge the rectangle with scale factor 2.
3. What are the dimensions of the image rectangle?
4. Calculate the alinear scale factor (L.S.F) of the enlargement.
5. Calculate the areas of the two rectangles and the ratio

   $\frac{\text{area of image rectangle}}{\text{area of object rectangle}}$

   This ratio is called the **area scale factor**.
6. How does this ratio compare with the linear scale factor?
7. Repeat your enlargement but with scale factors 3, $^{-}2$ and $\frac{1}{2}$. How do the new area scale factors compare with the corresponding linear scale factors?

Area scale factor (A.S.F) = (linear scale factor)^2.

$\therefore$ L.S.F = $\sqrt{\text{A.S.F}}$

Example 11.22

The ratio of the corresponding sides of two similar triangles is $\frac{3}{2}$. If the area of the smaller triangle is 6 cm^2, find the area of the larger triangle.

Solution

Since the two triangles are similar, the larger one is an enlargement of the smaller one.

$\therefore$ *linear scale factor = ratio of corresponding sides*

$= \frac{3}{2}$

Hence, area scale factor $= \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

Thus, area of larger triangle

= *area of smaller triangle × area scale factor*

$= 6 \times \frac{9}{4}$ *cm*^{2}

$= 13.5$ *cm*^{2}

Example 11.23

On a model house, the windows measure 1.2 cm × 3 cm. The area of the actual house is 10 000 times bigger than the area on the model. Find the dimensions of the windows on the actual house.

Solution

$A.S.F = 10\ 000$

$L.S.F = \sqrt{\text{A.S.F}}$

$= \sqrt{10000}$

$LSF = \frac{\text{length of image}}{\text{length of object}} = 100$

Length of image = (L.S.F) × length of object

$= 1.2$ *cm* $\times 100$

$= 120$ *cm* $= 1.2$*m*

and 3 *cm* × 100 = 300 *cm*

= 3.0 *m.*

$\therefore$ *Dimensions are 1.2m by 3.0m*

Exercise 11.10

1. A rectangle whose area is 18 cm^2 is enlarged with linear scale factor 3. What is the area of the image rectangle?
2. A pair of corresponding sides of two similar triangles are 5 cm and 8 cm long.
   (a) What is the area scale factor?
   (b) If the larger triangle has an area of 256 cm^2, what is the area of the smaller triangle?
3. The ratio of the areas of two circles is 16 : 25.
   (a) What is the ratio of their radii?
   (b) If the smaller circle has a diameter of 28 cm, find the diameter of the larger circle.
4. Two photographs are printed from the same negative. The area of one is 36 times that of the other. If the smaller photograph measures 2.5 cm by 2 cm, what are the dimensions of the larger one?
5. A lady found that a carpet with an area of 13.5 m^2 fitted exactly on the floor of a room 4.5 m long. If she moved the carpet to a similar room which is 1.5 m longer, how much floor area remained uncovered?
6. An architect made a model of a building to a scale of 1 cm : 2 m. A floor of the building is represented on the model with an area of 12 cm^2. What would be the corresponding area on another model of the same building whose scale is 2 cm : 1 m?
7. The scale of a map is 1:500 000. A section of sea has an area of 38.6 cm^2. Find the actual area of the sea represented on the map in hectares.
8. Small packets of tea leaves were packed in boxes measuring 30 cm by 42 cm by 24 cm for dispatch. Given that each packet was similar to the container and the ratio of sides of the box to the packet was 4 : 1, find the surface area of each packet of tea leaves.

11.2.7 Volume scale factor

Activity 11.18

Fig. 11.69 shows a cuboid and its enlargement with scale factor 2.

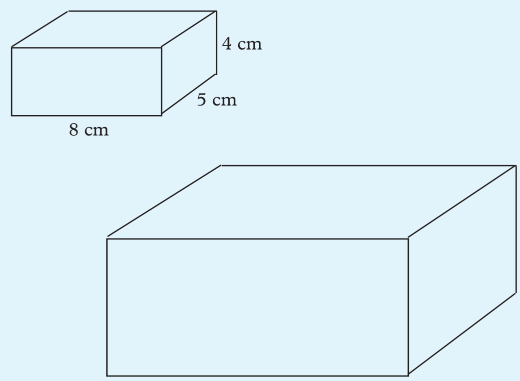


Fig. 11.69

1. What are the dimensions of the image?
2. Calculate the volumes of the two cuboids and the ratio

   $$\frac{\text{Volume of image cuboid}}{\text{Volume of object cuboid}}$$

   This ratio is called the **volume scale factor (V.S.F)**.
3. How does this ratio compare with the linear scale factor?
4. If instead the scale factor of enlargement is 3, what are the dimensions and volume of the image? How does the volume scale factor compare with the

linear scale factor now? If the scale factor of enlargement is $\frac{1}{2}$, what is the volume scale factor?

5. Given area scale factor, how would one get volume scale factor; and vice versa,

You should have discovered that:

Volume scale factor = (linear scale factor)^3

Since L.S.F = $\sqrt{ASF}$

Then, V.S.F = $(L.S.F)^3 = (\sqrt{A.S.F})^3$

Similarly

Since L.S.F = $\sqrt[3]{V.S.F}$

A.S.F = $(L.S.F)^2 = (\sqrt[3]{V.S.F})^2$

Note: For solids of the same material, the ratio of their masses is equal to the ratio of their volumes.

Example 11.24

A solid metal cone with a base radius 7 cm has a volume of 176 cm³. What volume would a similar cone made of the same metal and with base radius 10.5 cm have?

Solution

The ratio of radii is 10.5 : 7 = 3 : 2

i.e. linear scale factor $= \frac{3}{2}$

$\therefore$ volume scale factor $= \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

Hence, volume of larger cone

$= \frac{27}{8} \times$ volume of smaller cone,

$= \frac{27}{8} \times 176\ cm^3$

$= 594\ cm^3$

Note: *For solids of the same material, the ratio of their masses is equal to the ratio of their volumes.*

Example 11.25

Two similar cones A and B are such that the ratio of their volumes is 108:500. The smaller cone has a curved surface area of 504 cm². Find the area of the curved surface of the bigger cone.

Solution

$V.S.F = \frac{108}{500} = \frac{27}{125}$

$(L.S.F) = \sqrt[3]{V.S.F} = \sqrt[3]{\frac{27}{125}} = \frac{3}{5}$

$(A.S.F) = (L.S.F)^2 = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$

$\therefore$ Area of bigger cone $= \frac{25}{9} \times 54$

$= 150\ cm^2$

Curved surface area of bigger cone

$= 150\ cm^2$

Example 11.26

The areas of two similar solids is 49 cm² and 64 cm².

(a) *Find their volume scale factor.*

(b) *If the smaller one has a volume of 857.5 cm³, what is the volume of the larger one?*

Solution

Area of solid A = 49

Area of solid B = 64

$\therefore A.S.F = \frac{49}{64}$

$L.S.F = \sqrt{A.S.F} = \sqrt{\frac{49}{64}} = \frac{7}{8}$

$V.S.F = (L.S.F)^3 = \left(\frac{7}{8}\right)^3 = \frac{343}{512}$

$\frac{\text{Volume of small}}{\text{Volume of big}} \Rightarrow \frac{343}{512} = \frac{857.5\ cm^3}{v_b}$

$\Rightarrow v_b = \frac{512 \times 857.5\ cm^3}{343}$

$= 1\ 280\ cm^3$

Exercise 11.11

1. The volume scale factor of two similar solids is 27. What are their linear and area scale factors?
2. A cylinder with base radius r has a volume of 77 cm^3. What is the volume of a similar cylinder with base radius
   (a) 2r (b) 3r ?
3. Two similar jugs have capacities of 2 litres and 3 litres. If the height of the larger jug is 30 cm, find the height of the smaller one.
4. Two similar cones made of the same wood have masses 4 kg and 0.5 kg respectively. If the base area of the smaller cone is 38.5 cm^2, find the base area of the larger one?
5. A concrete model of a commemoration monument is 50 cm high and has a mass 30 kg.
   (a) What mass will the full size monument of height 9 m have if it is also made of concrete?
   (b) If the surface area of the model is 5 000 cm^2, what is the surface area of the full size monument?
6. Fig. 11.70 is a drawing of a model of a swimming pool. Find the capacity of the actual pool, in litres, given that its length is 27 m.

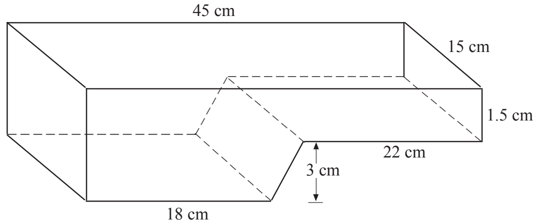


Fig. 11.70

7. The volume of two similar cones are 960 cm^3 and 405 cm^3. If the area of the curved surface of the bigger cone is 300 cm^2, find the surface area of the smaller cone.
8. The surface areas of two similar containers are 810 cm^2 and 1 440 cm^2. Find the linear scale factor of the containers. Hence, find the capacity of the larger container if the smaller one has a capacity of 108 litres.
9. Two similar square based pyramids have base areas of 9 cm^2 and 36 cm^2 respectively. Calculate
   (a) the height of the larger pyramid if the smaller pyramid is 9 cm high.
   (b) the ratio of the height to the width of the smaller pyramid if the ratio for the larger pyramid is 3 : 1.
   (c) the inclination of a slant face of the larger pyramid if for the smaller one is 76°.
   (d) the volume of the larger pyramid if the smaller one has a volume of 27 cm^3.

Unit Summary

- Two triangles are similar if the corresponding sides are in proportion, i.e. have a constant ratio or the corresponding angles are equal. Congruent triangles are also similar.
- For all shapes, other than triangles, both similarity conditions must be satisfied for them to be similar.
- The ratio of the corresponding sides in similar figures is called a ***linear scale factor***.

$$\text{L.S.F} = \frac{\text{Length of image}}{\text{Length of object}}$$

- For all solids, corresponding angles must be equal and the ratios of corresponding lengths must be equal for the solids to be similar.
- All cuboids are equiangular since all the faces are either rectangular or squares. However, not all cuboids are similar but all cubes are.
- If two figures are similar, the ratio of their corresponding areas equals the square of their linear scale factor.

  $\text{A.S.F} = (\text{L.S.F})^2$

- If two solids are similar, then the ratio of their corresponding volumes equals the cube of their linear scale factor.

  $\text{V.S.F} = (\text{L.S.F})^3$

- An enlargement is defined completely if the scale factor and the centre of enlargement are known.
- In an enlargement the object and its image are similar. If the scale factor is ±1, the object and its image are identical.

  Scale factor of enlargement

  $= \dfrac{\textit{Length of image}}{\textit{Length of object}}$

  $= \dfrac{\textit{Image distance from the centre of enlargement}}{\textit{Object distance from the centre of enlargement}}$
- To locate the centre of enlargement given an object and its image,
  (a) join any two pairs of corresponding points.
  (b) produce the lines until they meet at a point, that point is the centre of enlargement.
- An enlargement scale factor k centre origin (0, 0) maps a point P(a, b) onto P′(ka , kb).

Unit 11 test

1. In ΔABC (Fig. 11.71) identify two similar triangles in the figure and use them to find the values of a and b

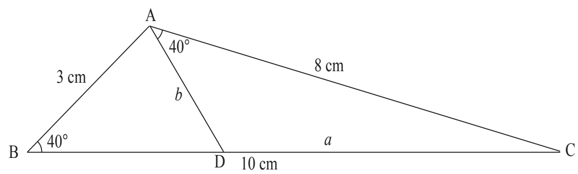


Fig. 11.71

2. ΔABC has coordinates A(–3, 1), B(–3, 4), C(–1, 4). ΔA′B′C′ is such that A′(–1, –3), B′(–1, 3) and C′(3, 3). Draw the two triangles on a graph paper. Find the centre and the scale factor of enlargement that maps ΔABC onto ΔA′B′C′.
3. Find the image of ΔABC A(2, 4), B(4, 2), C(5, 5), under enlargement centre (0, 0) scale factor 2 and state the coordinates of A′, B′ and C′, the images of points A, B and C.
4. The volumes of two similar cuboids are 500 cm^3 and 108 cm^3 respectively. Find the ratio of their
   (a) volumes, (b) surface areas.

   If the larger cuboid has a total surface area 400 cm^2, find the surface area of the smaller cuboid.
5. In ΔABC, BC = 5 cm, AC = 6 cm and ∠ACB = 49°. If ΔABC is enlarged to ΔA′B′C′ using a linear scale factor $\frac{3}{2}$,
   (a) write down the size of ∠A′C′B′ in relation to ΔABC
   (b) calculate the length of the side A′C′.
   (c) calculate the ratio of the area of ΔABC to the area of ΔA′B′C′ in relation to ΔABC.
   (d) find the ratio A′C′ : B′C′.
6. In the Fig. 11.72, ABCD is a trapezium with AB parallel to DC. The diagonals AC and BD intersect at E.
   (a) Giving reasons, show that ΔABE is similar to ΔCDE.
   (b) Given that AB = 3DC find the ratio DB to EB.

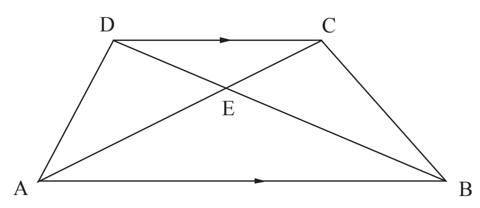


Fig. 11.72

7. A bowl in the shape of a hemisphere has a radius of length 10 cm. A similar bowl has a radius of length 20 cm. Calculate:
   (a) the circumference of the larger bowl if that of the smaller one is 64 cm.
   (b) the surface area of the larger bowl if that of smaller one is 629 cm^2.
   (c) the capacity of the larger bowl if that of the smaller one is 2.1 litres.

8. In Fig. 11.73, AB//DE. Identify two similar triangles and use them to find the values of x and y.

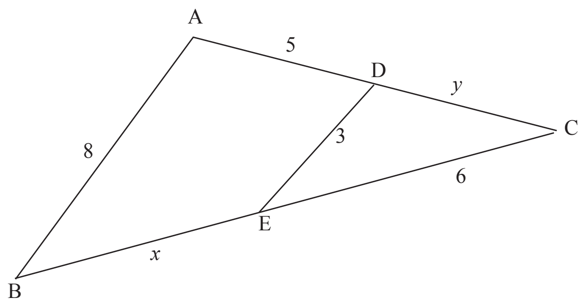


Fig. 11.73

12 INVERSE AND COMPOSITE TRANSFORMATIONS IN 2D

Key unit competence: By the end of this unit, learners should be able to solve shape problems involving inverse and composite transformations.

Unit outline

- Composite translations in two dimensions
- Composite reflections in two dimensions
- Composite rotations in two dimensions
- Mixed transformations in two dimensions
- Inverse transformations two dimensions.

Introduction

Unit Focus Activity

ABCD is a trapezium whose vertices are A(1 , 2), B(7 , 2), C(5 , 4) and D(3 , 4).

(a) On the same grid:

  (i) Draw ABCD and its image A′B′C′D′ under a rotation of 90° clockwise about the origin.

  (ii) Draw the image A″B″C″D″ of A′B′C′D′ under a reflection in the line $y = x$. State the co-ordinates of A″B″C″D″.

(b) A‴B‴C‴D‴ is the image of A″B″C″D″ under the reflection in the line $y = 0$. Draw the image A‴B‴C‴D‴ and state its co-ordinates.

(c) Describe a transformation that maps A‴B‴C‴D‴ onto ABCD.

We have also learnt how to perform single transformations including reflection, rotation and translation. In this unit, we will discover how to perform composite transformation i.e one transformation followed by another and so on.

12.1 Composite translations in two dimensions

Activity 12.1

Given the vertices of a triangle ABC as A (–4, 3), B (–1, 1) and C (–2, 5).

(a) Plot the points ABC on a Cartesian plane

(b) Join the points A, B and C to form triangle ABC

(c) Move point A horizontally to the right by 6 units and then vertically upwards by 2 units. Locate the final position of A as A′.

(d) Repeat procedure (c) for points B and C.

(e) Join the points A′, B′ and C′

(f) What do you observe about the size and shape of ABC and A′B′C′?

(g) Move point A′ 4 units downwards along the grid. Then locate the final position A″. Repeat this procedure for both points B′and C′.

Example 12.1

Triangle ABC is such that A (–4, 3), B (–1, 1) and C (–2, 5).

(a) *Find the co-ordinates of its image after a translation whose vector is* $\begin{pmatrix}6\\2\end{pmatrix}$.

(b) *The image of ABC in (a) above is then translated by the vector* $\begin{pmatrix}-2\\-5\end{pmatrix}$. *Find the co-ordinates of the new image.*

(c) *Plot ABC and its images on the same Cartesian plane.*

Solution

(a) $\mathbf{OA'} = \begin{pmatrix}-4\\3\end{pmatrix} + \begin{pmatrix}6\\2\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix}$, A′(2,5)

$O\mathbf{B'} = \begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}6\\2\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix}$, B′(5,3)

$O\mathbf{C'} = \begin{pmatrix}-2\\5\end{pmatrix} + \begin{pmatrix}6\\2\end{pmatrix} = \begin{pmatrix}4\\7\end{pmatrix}$, C′(4,7)

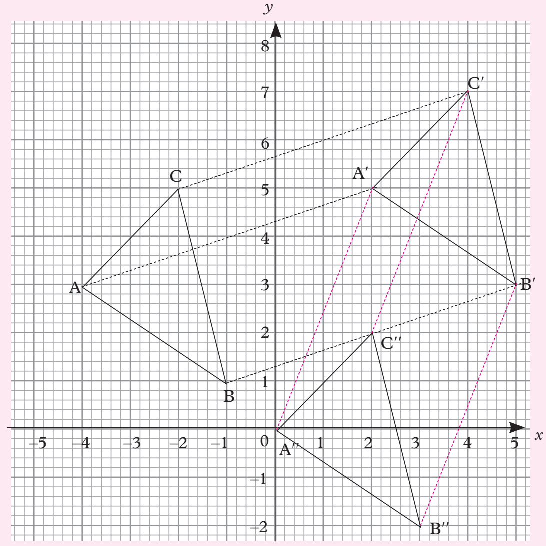
(b) $O\mathbf{A''} = \begin{pmatrix}2\\5\end{pmatrix} + \begin{pmatrix}-2\\-5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, A″(0,0)

$\mathbf{OB''} = \begin{pmatrix}5\\3\end{pmatrix} + \begin{pmatrix}-2\\-5\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$, B″(3,-2)

$\mathbf{OC''} = \begin{pmatrix}4\\7\end{pmatrix} + \begin{pmatrix}-2\\-5\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$, C″(2,2)

(c) *Therefore the co-ordinates of the images are: A′ (2, 5), B′ (5, 3) and C′ (4, 7) and A″ (0, 0), B″ (3,–2), C″ (2, 2).*

This can be illustrated on the Cartesian plane as shown in graph 12.1.



Graph 12.1

Composite transformation takes place when two or more transformations combine to form a new transformation. Here, one transformation produces an image upon which the other transformation is then performed.

In a translation, all the points in the object move through the same distance and in the same direction. In order to obtain the image of a point under a translation, we add the translation vector to the position vector of the point.

Example 12.2

Points A(1,1), B(1,2), C(4,2) and D(4, 1) represent a rectangle.

(a) *Translate ABCD under vector* $\begin{pmatrix}-2\\-1\end{pmatrix}$. *Write down image co-ordinates A′, B′, C′ and D′.*

(b) *Translate A′, B′, C′ and D′ under vector* $\begin{pmatrix}2\\2\end{pmatrix}$. *Write down image co-ordinates for* A″B″C″ *and* D″.

Solution

(a) $\mathbf{OA'} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}-2\\-1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

$\mathbf{OB'} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-2\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

$\mathbf{OC'} = \begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}-2\\-1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

$\mathbf{OD'} = \begin{pmatrix}4\\1\end{pmatrix} + \begin{pmatrix}-2\\-1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$

$\therefore$ *A′(–1,0), B′$\begin{pmatrix}1\\1\end{pmatrix}$, C′(2, 1) and D′(2, 0).*

(b) $\mathbf{OA''} = \begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$OB'' = \begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$

$OC'' = \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}$

$OD'' = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$

$\therefore$ *A″(1,2), B″(1,3), C″(4,3) and D″(4, 2).*

Exercise 12.1

1. (a) Triangle XYZ is such that X(–2, 1), Y(–3, 2) and Z(–3, 4). It is given a translation of vector $\begin{pmatrix}2\\2\end{pmatrix}$. Find the co-ordinates of its image, X′Y′Z′.
   (b) The image X′Y′Z′ is translated by the vector $\begin{pmatrix}-4\\2\end{pmatrix}$. Find the new image X″Y″Z″.
   (c) Plot XYZ and its images X′Y′Z′ and X″ Y″Z″ on the same Cartesian plane.
2. The points A (4, 0), B (4, 2), C (0, 2) and D (0, 0) represent a rectangle.
   (a) Find the image A′B′C′D′ of ABCD under translation represented by vector $\begin{pmatrix}-3\\4\end{pmatrix}$.
   (b) The image of ABCD is further translated by vector $\begin{pmatrix}8\\0\end{pmatrix}$. Find the co-ordinates of the new image A″B″C″D″.
   (c) Plot the ABCD and its images on the same Cartesian plane.
3. Points A (3, 4), B (3, 7), C (6, 7) and D (9, 4) represent a trapezium.
   (a) Find the image A′B′C′D′ of trapezium ABCD under a translation represented by vector $\begin{pmatrix}-6\\-5\end{pmatrix}$.
   (b) The Image obtained in (a) above is then translated by vector $\begin{pmatrix}3\\-2\end{pmatrix}$. What are the co-ordinates of the new image A″B″C″D″?
   (c) Plot ABCD and its images on the same Cartesian plane.
4. Triangle PQR has co-ordinates P(0, 0), Q(–3, 2) and R′(2,2).
   (a) What is the image P′Q′R′ of PQR under a translation represented by the vector $\begin{pmatrix}-2\\-2\end{pmatrix}$?
   (b) The image obtained in (a) above is further translated by the vector $\begin{pmatrix}-4\\5\end{pmatrix}$. What are the co-ordinates of the new image P″Q″R″?
   (c) Plot PQR and its two images on the same Cartesian plane.
5. Rectangle ABCD has vertices A(–1, 0), B(–1, 2), C(1, 0) and D(1, 2).
   (a) Draw rectangle ABCD on a Cartesian plane.
   (b) The rectangle is translated by the column vector $\begin{pmatrix}-4\\5\end{pmatrix}$. Write down the co-ordinates of the image A′B′C′D′ and draw it on the

same Cartesian plane with the object ABCD.

(c) The image A′B′C′D′ is also translated by the column vector $\begin{pmatrix}2\\-3\end{pmatrix}$. Write down the co-ordinates of the second image A″B″C″D″. Draw A″B″C″D″ on the same Cartesian plane.

12.2 Composite reflections in two dimensions

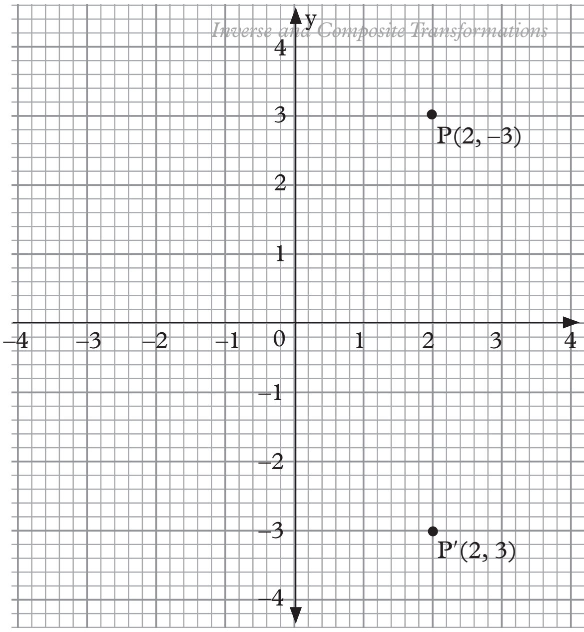
Activity 12.2

1. Given the vertices of triangle ABC as A(-2, 1), B(-5,1) and C(-4,-3).
   (a) Draw ABC, line x=0 and line y=3 on the Cartesian plane.
   (b) Reflect triangle ABC into the line x=0 to form the image A′B′C′. Write down the co-ordinates of the image.
   (c) Reflect A′B′C′ into the line y=3 to k form the image A″B″C″. Write down the co-ordinates of the image.

A composite reflection is two or more reflection performed one after the other. Some simple reflections can be performed easily in the co-ordinate plane using the following general rules.

(a) Reflection in x-axis

Consider point P(2, 3) in the graph and assume x-axis is the mirror. The image of P under the reflection is P′(2, –3). The x-axis value does not change but the y-axis value changes the direction.

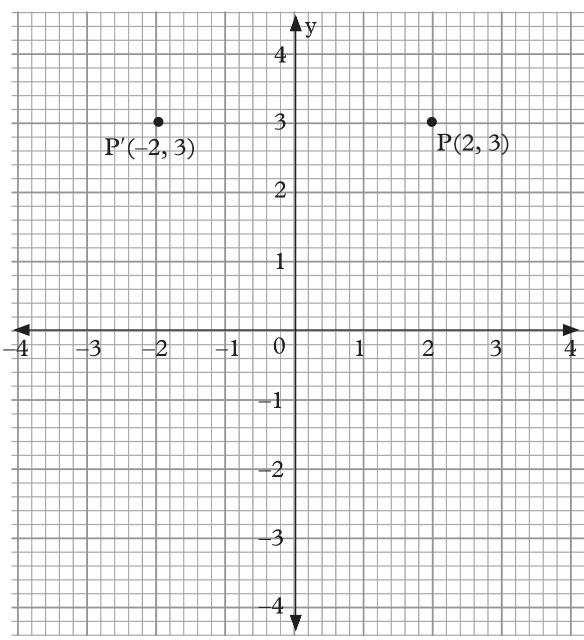


Graph 12.3

The rule for the reflection in the x-axis is $P(x, y) \to (x, -y)$.

(b) Reflection in the y-axis

Consider point in the graph 12.4 and assume axis is the mirror. The image of P(–2, 3). The y-axis value does not change but the y-axis value changes the direction.

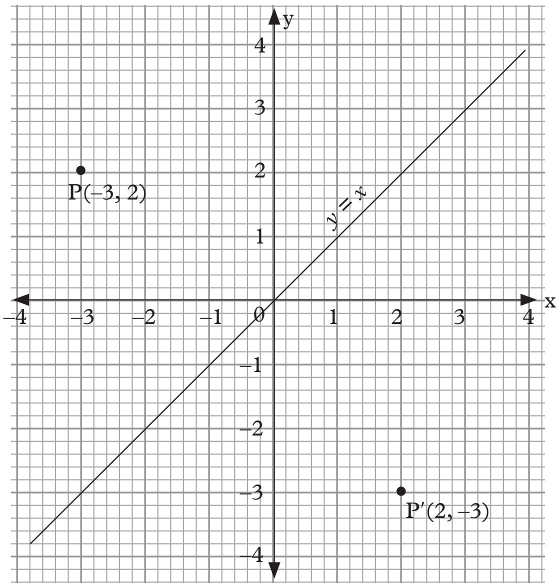


Graph 12.4

The rule for the reflection in the y-axis is $P'(x, y) \rightarrow P'(-x, y)$.

(c) Reflection in line y = x

Consider point P(–3, 2) in the graph12.5 below and $y = x$ assume is the mirror. The image of P under the reflection is P′(2, –3) the -axis value becomes -axis value while-axis value becomes-axis value.

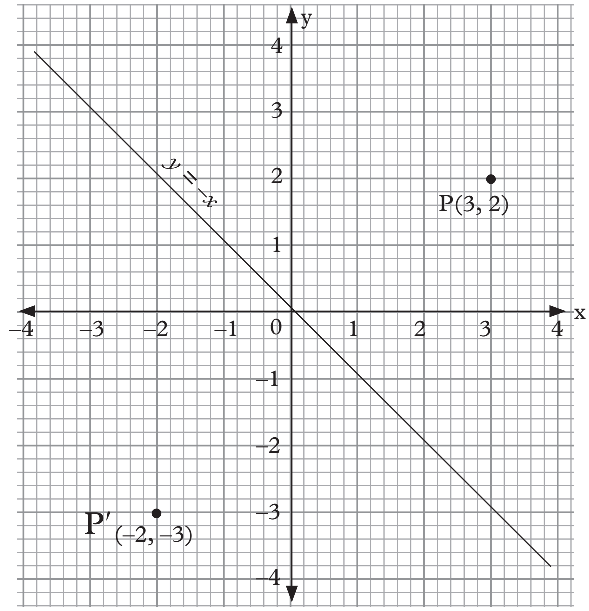


Graph 12.5

The rule for the reflection in the line is $y = x$ is $P(x, y) \rightarrow P'(y, x)$.

(d) Reflection in the line y = –x

Consider point P in the graph 12.6 and assume line $y = -x$ is the mirror. The image of P (3, 2) under the reflection is the x-axis value becomes $-y$-axis value while y-axis value becomes $-x$-axis value ie P′ (–2, –3).



Graph 12.6

The rule for a reflection in the line $y = -x$ is $P(x, y) \rightarrow P'(-y, -x)$.

(e) Equation of the mirror line

Equation of the mirror line can be found by using the coordinates of the midpoints of the image and the object lines.

Let us consider the object whose coordinates are $A(a, b)$, $B(c, d)$ and the image $A'(a_1, b_1)$, $B'(c_1, d_1)$, The equation of the mirror line is got from the midpoints of A, A′ and B, B′ as;

$$M_1\left(\frac{a + a_1}{2}, \frac{b + b_1}{2}\right) \text{ and } M_2\left(\frac{c + c_1}{2}, \frac{d + d_1}{2}\right).$$

The equation of the line joining points M_1 and M_2 is the equation of the mirror line.

Example 12.3

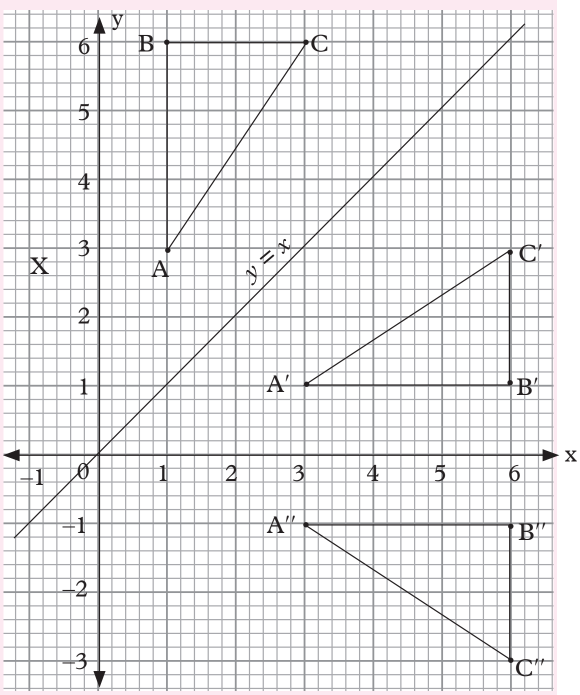
The points A (1, 3), B (1, 6) and C (3, 6) are the vertices of triangle ABC.

(a) *On a Cartesian plane, draw triangle ABC.*

(b) *Triangle ABC is reflected in the line $y = x$. Draw the image A'B'C' on the same Cartesian plane and state its co-ordinates.*

(c) *Triangles A'B'C' is then reflected in the line $y = 0$ (x-axis). Draw the image A''B''C'' of A'B'C' on the same Cartesian plane and state its co-ordinates.*

Solution



Graph 12.7

Co-ordinates of the images.

A' (3,1), B' (6,1), C' (6,3)

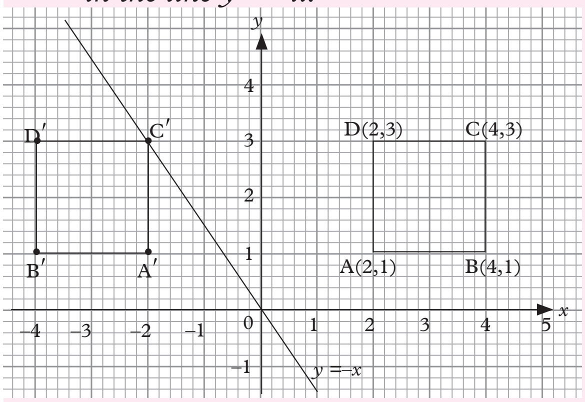
A'' (3,–1), B'' (6,–1), C'' (6, –3)

Example 12.4

A quadrilaterial ABCD has vertices A(2,1), B(4, 1), C(4, 3) and D(2, 3).

(a) *Find the image of the quadrilaterial ABCD under a reflection in the line $x = 0$.*

(b) *State its co-ordinates under a reflection in the line $y = –x$.*



Graph 12.8

Example 12.5

The vertices of a quadrilateral are A(2,0.5), B(2,2), C(4,3.5) and D(3.5, 1). Find the image of the quadrilateral under reflection in line $y=0$ followed by reflection in line $y = –x$.

Solution

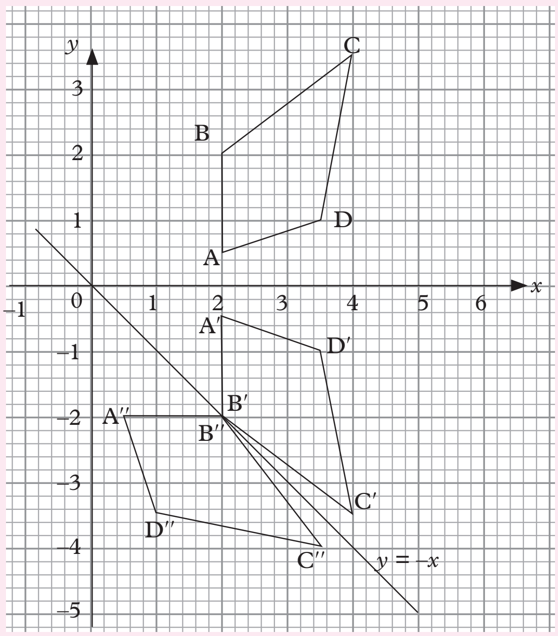
'Reflection in line $y=0$ followed by reflection in line $y=–x$ means that we first obtain the image under reflection in line $y=0$ and then reflect this image in line $y = –x$.

This is shown in graph 12.9. In the graph, A'B'C'D' is the reflection of ABCD in line $y = 0$. A''B''C''D'' is the reflection of A'B'C'D' in line $y = –x$.

Thus the required image vertices are:

A' (2, –0.5), B' (2, –2), C' (4, –3.5) and D' (3.5, 1)

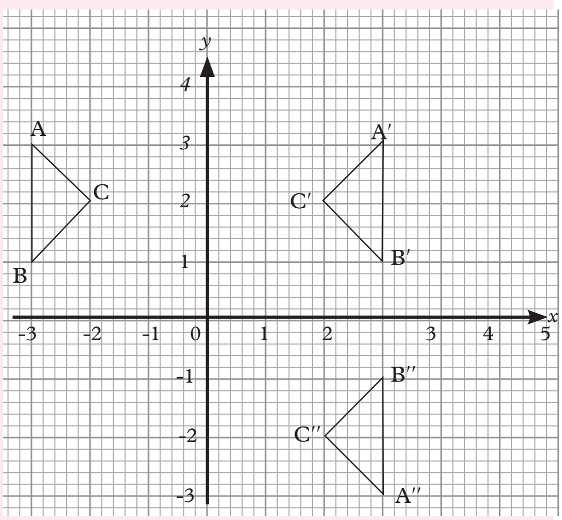
A'' (0.5, –2), B'' (2, –2), C'' (3.5, –4) and D'' (1, 3.5)



Graph 12.9

Example 12.6

In graph 12.10 triangle ABC has vertices A(–3,3), B(–3,1) and C(–2,2). A′B′C′ and A″B″C″ are its images after given reflections.



Graph 12.10

(a) *Write down all image co-ordinates.*

(b) *Write down equations of the reflection lines.*

Solution

(a) *Image co-ordinates.*

A′(3,3), B′ (3,1), C′(2,2)

A″(3,–3) B″(3,–1), C″(2,–2)

(b) *Equations of refrection lines.*

From triangle ABC to A′B′C′, equation of the line x=0.

From triangle A′B′C′ to A″B″C″, equation of the line is y=0.

Exercise 12.2

1. (a) Draw triangle ABC with vertices as A (6, 8), B (2, 8), C (2, 6). Draw the lines $y = 2$ and $y = x$.
   (b) Draw the image of triangle ABC after reflection in;
      (i) The y-axis. Label it triangle A′B′C′.
      (ii) The x-axis. Label it triangle A″B″C″.
      (iii) The line $y=x$. Label it triangle A‴B‴C‴.
   (c) Write down the co-ordinates of the image of point A in each case.
2. (a) Plot triangle 1 defined by (3,1), (7,1) (7,3).
   (b) Reflect triangle 1 in the line $y = x$ onto triangle 2.
   (c) Reflect triangle 2 in the line $y = 0$ (x-axis) onto triangle 3.
   (d) Reflect triangle 3 in the line $y = -x$ onto triangle 4.
   (e) Reflect triangle 4 in the line $x = 2$ onto triangle 5.
   (f) Write down the co-ordinates of triangle 5.

3. (a) Construct triangle 1 at (2, 6), (2, 8), (6, 6).
   (b) Reflect triangle 1 in the line $x + y = 6$ onto triangle 2
   (c) Reflect triangle 2 in the line $x = 3$ onto triangle 3.
   (d) Reflect triangle 3 in the line $x + y = 6$ onto triangle 4
   (e) Reflect triangle 4 in the line $y = x - 8$ onto triangle 5.
   (f) Write down the co-ordinates of triangle 5.
4. (a) Draw and label the following triangles.
   Triangle 1: (3, 3), (3, 6), (1, 6)
   Triangle 2: (3,–1), (3, –4), (1, –4)
   Triangle 3: (3, 3), (6, 3), (6, 1)
   Triangle 4: (–6, –1), (–6, –3), (–3, –3)
   Triangle 5: (–6, 5), (–6, 7), (–3, 7)

   (b) Find the equation of the mirror line for the reflection;
   (i) Triangle 1 onto triangle 2
   (ii) Triangle 1 onto triangle 3
   (iii) Triangle 1 onto triangle 4
   (iv) Triangle 4 onto triangle 5.

5. Rectangle ABCD has vertices A(–1,0), B(–1,2), C(1,0) and D(1,2).
   (a) Construct rectangle ABCD on a Cartesian plane.
   (b) Reflect rectangle ABCD in the x-axis.
   (i) Write down the image coordinates.
   (ii) Construct the image of ABCD on the same Cartesian plane.

12.3 Composite rotations in two dimensions

Activity 12.3

1. You are given a line segment with end points A (2, 2) and B (2, 5).
   (a) Draw line AB on a Cartesian plane.
   (b) Fix point A and rotate point B through 60° in clockwise direction.
   (c) Write down the co-ordinate B′ the image of B.
   (d) Leaving A as a fixed point, rotate point B′ 90° in anticlockwise direction.
   (e) Write down the co-ordinate B″ the image of B′.
2. Fig.12.1 shows unmarked clock.

Fig.12.1

   (a) Mark the clock with appropriate time values if the minute and hour hands are moving in clockwise direction.
   (b) Mark the clock with appropriate time values if the minute and hour hands are moving in anti-clockwise direction.

The three important factors of rotation are direction of rotation, Centre of rotation and the angle of rotation.

The angle of rotation is measured in degrees. A rotation of 360^0 is called a

revolution. A rotation of 270° is the equator turn. A rotation of 180° is called a **half turn** and a rotation of 90° is called a **quarter turn**. Figure 12.11 shows the various turns. The dotted lines show the initial positions.

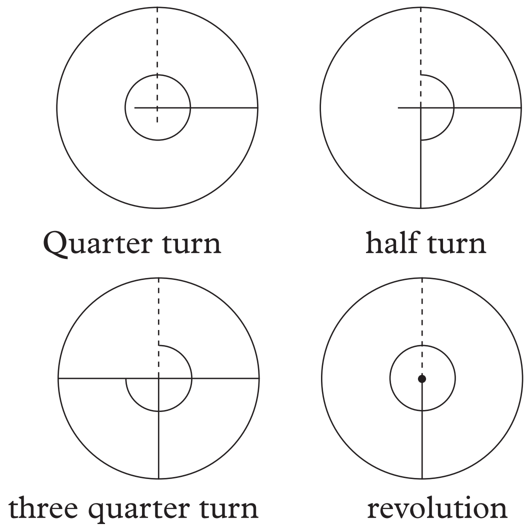


Fig. 12.2

The direction in which the hands of a clock turn is called **clockwise** and the rotation in the opposite direction is called **anticlockwise**.

The angle of rotation in an anticlockwise direction is **positive** and that in a clockwise direction is **negative**.

Thus, a rotation of 90° anticlockwise is written as +90° and 90° clockwise as -90°.

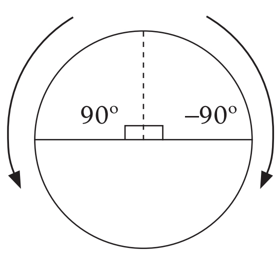


Fig. 12.3

When an object is given a rotational transformation, the image is always the same size as the object. Such a transformation is called an **isometry**.

If M(h,k) is any point, its rotation follows the following rules.

1. M(h,k) has image M′($-k$,h) after rotation at 90° in anticlockwise direction.
2. M(h,k) has the image M′(k,$-h$) after rotation at 90° in clockwise direction.
3. M(h,k) has the image M′($-h$, $-k$) after rotation at 180° in either clockwise or anticlockwise direction.

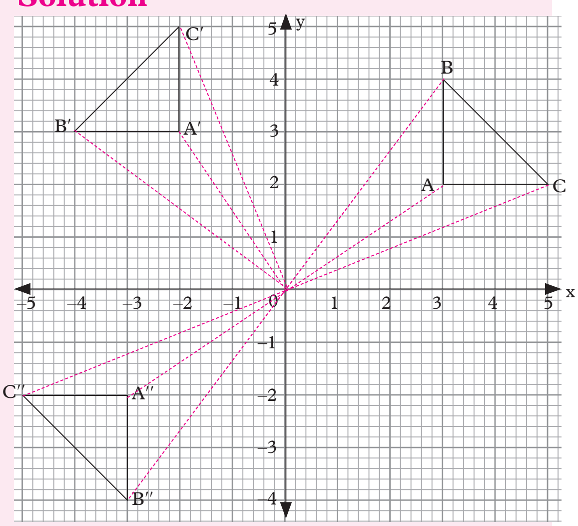
Example 12.7

Given the vertices of triangle ABC as A (3, 2), B (3, 4) and C (5, 2);

(a) Rotate triangle ABC through +90° about the origin onto A′B′C′. Write down the co-ordinates of A′B′C′

(b) Rotate A′B′C′ through + 90° about the origin onto .Write down the coordinates of A′′B′′C′′

Solution



Graph 12.10

Co-ordinates of the Images.

A′(–2, 3), B′ (–4, 3), C′(–2, 5)

A′′(–3,–2), B′′ (–3,–4), C′′ (–5,–2)

Example 12.8

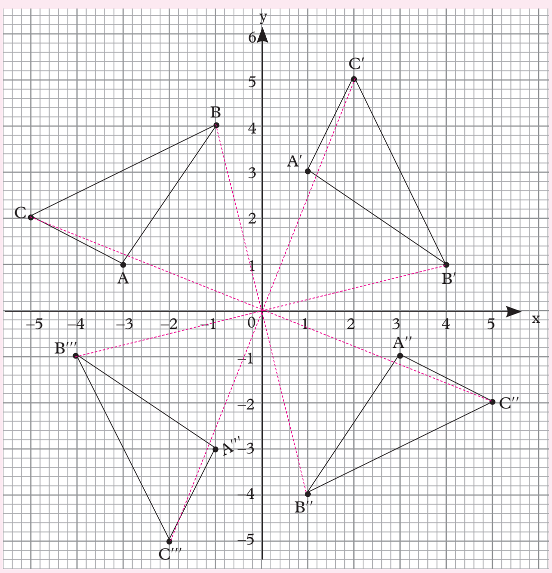
Consider triangle A(–3, 1), B(–1, 4), and C(–5, 2).

(a) *Rotate triangle ABC through –90° about the origin onto A′, B′, C′.*

(b) *Rotate A′B′C′ through –90° about the origin onto triangle A″B″C″.*

(c) *Rotate A″B″C″ through –90° about the origin to get triangle A‴B‴C‴. Write down the co-ordinates of images A′B′C′ , A″B″C″ and A‴B‴C‴.*

Solution



Graph 12.11

(d) *A′(1, 3), B′(4, 1), C′(2, 5)*
A″(3, –1), B″(1, –4) C″(5, –2)
A‴(–1, –3), B‴(–4, –1),
A‴(–1, –3) C‴(–2, –5),

Exercise 12.3

1. (a) Draw triangle one at A(1,2), B(1,6), C(3,5).
   (b) Rotate triangle one 90° clockwise centre (1,2) onto triangle two.
   (c) Rotate triangle two 180° centre (2,-1) onto triangle three.
   (d) Rotate triangle three 90° clockwise, centre (2,3) onto triangle four
   (e) Write down the co-ordinates of triangle four.
2. (a) Draw and label the following triangles on a Cartesian plane.
   Triangle 1: (3, 1), (6, 1), (6, 3)
   Triangle 2: (-1, 3), (-1, 6), (-3, 6)
   Triangle 3: (1, 1), (-2, 1), (-2,-1)
   Triangle 4: (-3, 1), (–3,4), (–5,4)
   (b) Describe fully the following rotations.
   (i) 1 onto 2 (ii) 1 onto 3
   (iii) 1 onto 4 (iv) 3 onto 2
3. (a) Draw triangle 1 at (4,7), (8,5), (8,7).
   (b) Rotate triangle 1, 90° clockwise centre (4, 3) onto triangle 2.
   (c) Rotate triangle 2, 180° centre (5, –1) onto triangle 3.
   (d) Rotate triangle 3, 90° anticlockwise, centre (0, –8) onto triangle 4.
   (e) Describe fully the following rotations;
   (i) Triangle 4 onto triangle 1
   (ii) Triangle 4 onto triangle 2.
4. (a) Draw triangle LMN at L(3, 1), M(6, 1), N(3, 3) and its image after half turn about the origin. State the co-ordinates of the vertices L′, M′ and N′ of the image of a triangle.
   (b) Rotate L′M′N′ of (a) above through +90° about the origin. State the co-ordinates of L″, M″, N″.

12.4 Mixed composite transformations in two dimensions

Activity 12.5

Given triangle ABC has vertices A(0, 2), B(2, 2) and C(2, 5).

(a) Plot triangle ABC on a Cartesian plane.

(b) Reflect triangle ABC in the line $x = 3$ to get the image A′,B′,C′. Write the co-ordinates of A′B′C′.

(c) Translate triangle A′B′C′ by the vector $\binom{2}{2}$ to get the image A″B″C″. Write down the co-ordinates of A″, B″, and C″.

An object can undergo several different transformations, one after the other. This is done such that the image of the preceding transformation becomes the object of the next transformation.

When an object, A, undergoes a transformation **R** followed by another transformation **N**, this is represented as **N**[**R** (A)] or **NR** (A).

The first transformation is always indicated to the right of the second transformation.

RR (A) means 'perform transformation R on A and then perform R on the image'. It also written R^2(A).

Example 12.9

A is a triangle with vertices (2, 1), (2, 4) and (4, 1). **R** *is a rotation of 90°.*

Clockwise about the origin and **N** *is a reflection in the y-axis. Find the vertices of the image of A if it undergoes the following transformations:*

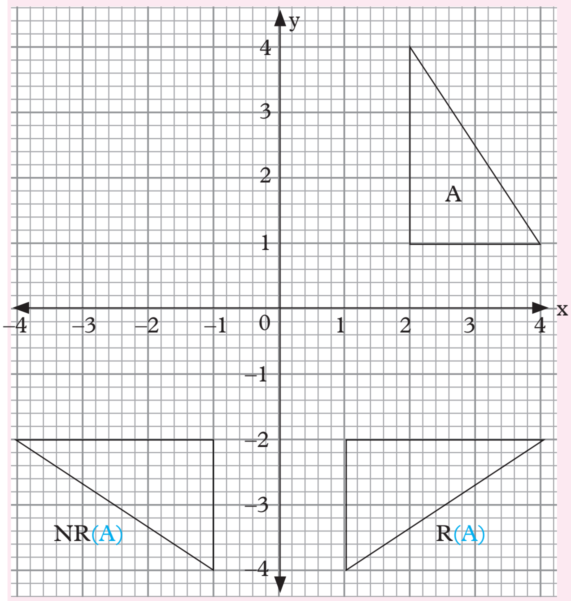
(a) **NR** *(A)* *(b)* **RN** *(A)*

Solution

The graph 12.12 shows triangle A under these transformations and its image **NR***(A) and* **R** *(A).*

(a) (2,1) (2,4) and (4,1)

(b) The final image of A has vertices (–1, –2), (–1, –4) and (–4, –2).

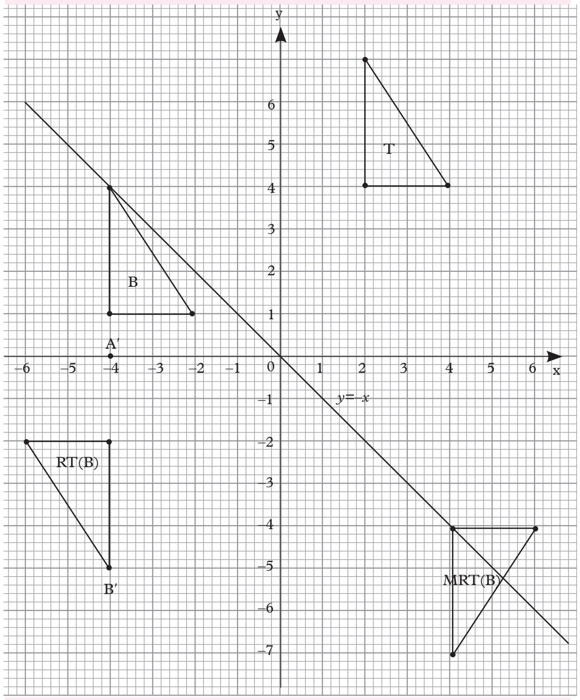


Graph 12.12

Example 12.10

Triangle B has vertices at (–2, 1), (–4, 1) and (–4, 4). **T** *is a translation represented by the vector* $\binom{8}{3}$, **R** *is a reflection in the line* $y = -x$ *and* **M** *is a rotation of 90° anticlockwise about the origin. Find the vertices of* **MRT***(B).*

Solution



Graph 12.13

Vertices of **MRT (B)** *are (4,-4) (6,-4) (4,-7).*

Example 12.11

Consider the triangle ABC with vertices A(–3, 5), B(–5, 1), C(–1, 3).

(a) *Translate triangle ABC onto triangle A′B′C′ under vector* $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

Write down co-ordinates of A′B′C′.

(b) *Reflect triangle A′B′C′ onto the line x=0 to get A″B″C″. What are the co-ordinates of A″B″C″.*

Solution

(a) $\mathbf{OA'} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$,

A′ (–4, 3)

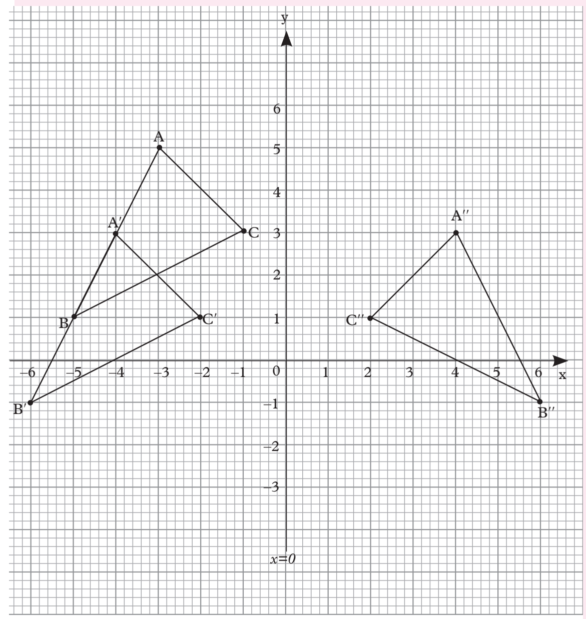
$\mathbf{OB'} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \end{pmatrix}$,

B′ (–6, –1)

$\mathbf{OC'} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$,

C′ (–2, 1)

(b) A″ (4, 3), B″ (6, –1), C″ (2, 1)



Graph 12.14

Exercise 12.4

1. Draw x- and y-axes with values from -8 to +8 and plot the point P (3, 2).

   R denotes 90° clockwise rotation about (0, 0);
   X denotes reflection in x = 0.
   H denotes 180° rotation about (0, 0);
   T denotes translation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

   For each question, write down the co-ordinates of the final image of P.

   (a) **R** (P)
   (b) **T** (P)
   (c) **TH** (P)
   (d) **HX** (P)
   (e) $\mathbf{R}^2$(P)
   (f) **TR** (P)
   (g) **RT** (P)
   (h) **XT** (P)
   (i) **XX** (P)
   (j) **XTH (P)**

From questions 2-6, use transformations **A, B, C, D, E,** as follows:

A denotes reflection in $x = 2$

B denotes 180^0 rotation, centre (1, 1)

C denotes translation $\begin{pmatrix} -6 \\ 2 \end{pmatrix}$

D denotes reflection in the line $y = x$.

E denotes reflection in $y = 0$

F denotes translation $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

G denotes 90° rotation clockwise, centre (0, 0)

2. Draw the axes with values from -8 to +8.

   Draw triangle LMN at L (2, 2), M (6, 2), N (6, 4). Find the image of LMN under the following combinations of transformations. Write down the co-ordinates of the image of point L in each case:
   (a) **CA** (LMN)
   (b) **ED** (LMN)
   (c) **DB** (LMN)
   (d) **BE** (LMN)
   (e) **EB** (LMN)
3. Draw triangle PQR at P (2, 2), Q (6, 2), R (6, 4). Find the image of PQR under the following combinations of transformations. Write down the co-ordinates of the image of point P in each case.
   (a) **AF**(PQR)
   (b) **CG**(PQR)
   (c) **AG**(PQR)
   (d) **AE**(PQR)
4. Draw triangle XYZ at X (-2, 4), Y (-2, 1),Y (-4, 1). Find the image of XYZ under the following combinations of transformations and in each case state the equivalent single transformation.
   (a) $\mathbf{G^2E}$(XYZ)
   (b) **CB**(XYZ)
   (c) **DA** (XYZ)
5. Draw triangle OPQ at O (0, 0), P (0, 2), Q (3, 2). Find the image of OPQ under the following combinations of transformations and state the equivalent single transformation in each case:
   (a) **DE**(OPQ)
   (b) **FC**(OPQ)
   (c) **DEC**(OPQ)
   (d) **DFE**(OPQ)
6. Draw triangle XYZ at X (1, 2), Y(1, 6), Z (3, 6).
   (a) Find the image of XYZ under each of the transformations **BC** and **CB**.
   (b) Describe fully the single transformation equivalent to **BC**.
   (c) Describe fully the transformation **M** such that **MCB** = **BC**.

12.5 Inverse transformations in two dimensions

Activity 12.5

You are given co-ordinates of figure ABCD as A(0, 0), B(2, 2), C(–2, 2) and D(–2, 0).

(a) On a Cartesian plane, draw the figure ABCD.

(b) Rotate figure ABCD through 90° about point (0, 0) in clockwise direction to get the image A′B′C′D′.

(c) Translate the image A′B′C′D′ with $T = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ to get image A″B″C″D″.

(e) Translate A″B″C″D″ with $T = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ to get the Image A‴B‴C‴D‴.

Plot all the images on the same Cartesian plane.

The inverse of a transformation reverses the transformation, i.e. it is the transformation which maps the image back to the object.

The following are examples of inverse transformations are defined by the following cases.

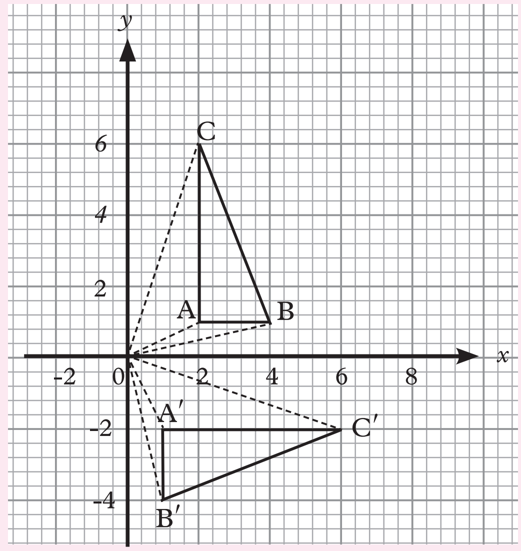
1. If $\boldsymbol{R}$ denotes θ° clockwise rotation, about (0, 0), then $\boldsymbol{R}^{-1}$ denotes θ° anticlockwise rotation about (0, 0).
2. If translation $\boldsymbol{T}$ has vector, the translation which has the opposite effect has vector. This is written as $\boldsymbol{T}^{-1}$
3. For all reflections, the inverse is the same reflection. For example if $\boldsymbol{X}$ is reflection then $\boldsymbol{X}^{-1}$ is also reflection.

   The symbol $\boldsymbol{T}^{-3}$ means $(\boldsymbol{T}^{-1})^3$ i.e. perform $\boldsymbol{T}^{-1}$ three times.

Example 12.12

An image of a triangle with vertices A′(1, –2), B′(1,–4) and C′(6, –1) is formed after an object ABC undergoes a clockwise rotation through 90° about the origin. Find by construction the co-ordinates of its object, ABC.

Solution



Graph 12.15

A(2, 1), B(4,1) and C(2,6)

Example 12.13

P′Q′R′S′ *is the image trapezium whose vertices are* P′(1,3), Q′(4,3), R′(1,1) and S′(6,1).

P′Q′R′S′ *is obtained after reflection in the x=0.*

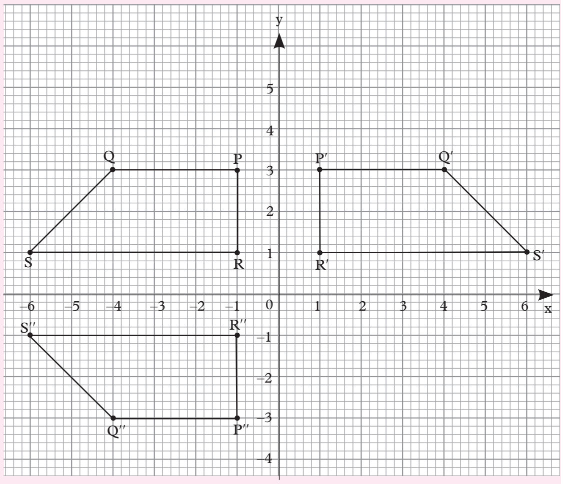
(a) *Obtain the co-ordinates of object trapezium PQRS.*

(b) *Write down the co-ordinates of* P″Q″R″S″ *the image of PQRS when reflected in the line y=0.*

Solution

All the co-ordinates can be obtained from graph 12.17 below.

- *The line x=0 is y–axis.*
- *The line y=0 is x*–axis.



Graph 12.16

P(–1, 3), Q(–4,3), R(–1,1) and S(–6,1).
P''(–1,–3), Q''(–4,–3), R''(–1,–1), and S''(–6,–1).

Exercise 12.5

Draw x- and y-axes with values from –8 to +8 and plot the point Q (3, 2).

R denotes 90° clockwise rotation about (0, 0);

X denotes reflection in $x = 0$.

H denotes 180° rotation about (0, 0);

T denotes translation $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Perform each of the following transformation. Write down the co-ordinates of the final image of Q.

1. $R^{-1}(Q)$
2. $T^{1}(Q)$
3. $(T^{3})\,H\,(Q)$
4. $HX^{-1}(Q)$
5. $RR^{-1}(Q)$
6. $T^{-1}R\,(Q)$
7. $R^{-1}T\,(Q)$
8. $XT^{1}(Q)$
9. $XX^{-1}(Q)$
10. $XT^{-1}H\,(Q)$

Unit Summary

- In composite translation, all the points in the object move through the same distance and in the same direction
- **Composite reflection**: A reflection is a trans-formation representing a flip of a figure.

  A reflection maps every point of a figure to an image across a fixed line. The fixed line is called the line of reflection
- **Equation of a mirror line:** Equation of the mirror line can be found by using the co-ordinates of the midpoints of the image and the object.

  For example the object whose co-ordinates are A(a, b) and B(c, d), and the image A'(a_1,b_1), B'(c_1,d_1), The equation of the mirror line is got from the midpoints of A, A' and B, B' as;

  $M_1\left(\frac{a + a_1}{2}, \frac{b + b_1}{2}\right)$ and

  $M_2\left(\frac{c + c_1}{2}, \frac{d + d_1}{2}\right)$.

  The equation of the line joining points M_1 and M_2 is the equation of the mirror line.
- **Composite rotation:** The turning of an object about a fixed point or axis is called **rotation.** The amount of turning is called the **angle of rotation**.
- **Mixed transformations:** An object can undergo several transformations, one after the other. This is done such that the image of the preceding transformation becomes the object of the next transformation.
- **Inverse of a transformation:** The inverse of a transformation reverses the transformation, i.e. it is the transformation which takes the image back to the object.

Unit 12 Test

1. Draw triangle XYZ with vertices at X (–2, 4), Y(–2, 1), Z (–4, 1). Find the image of XYZ under the following combinations of transformations.
   (a) A reflection in the line x = 0.
   (b) A rotation through an angle of 180° about (0, 0).
2. Plot triangle PQR at P(2, 2), Q(6, 2), R(6, 4). Find the image of PQR under the following combinations of transformations; a translation, under vector $\binom{2}{4}$ followed by a reflection in the line x=3. Write down the co-ordinates of the final image of point P.
3. **T** denotes translation $\binom{3}{2}$, **R** denotes 90° clockwise rotation about (0, 0);
   (a) Sketch x- and y-axes with values from -7 to +7 and plot the point Q (3, 2).
   (b) Locate the images of the following:
      (i) $\boldsymbol{R^{-1}}$ (Q)
      (ii) $\boldsymbol{T^{-1}R}$ (Q)
      (iii) $\boldsymbol{T^{1}}$ (Q)
4. (a) Plot the triangle ABC at A (4, 6), B (1, 6), C (1, 4). Draw the line y = 2 and y = x.
   (b) Plot the image of triangle ABC after reflection in;
      (i) The y-axis. Label it triangle 1
      (ii) The line y = 2. Label it triangle 2.
      (iii) The line y = x. Label it triangle 3.
   (c) Write down the co-ordinates of the image of A in each case.

13 STATISTICS (BIVARIATE DATA)

Key unit competence: By the end of this unit, the learner should be able to collect, represent and interpret bivariate data.

Unit outline

- Definition and examples bivariate data.
- Frequency distribution table of bivariate data.
- Review of data representation using graphs.
- Definition of scatter diagrams.
- Correlation
- Unit summary
- Unit Test

Introduction

Unit Focus Activity

In a certain school, 15 boys took two examination papers in the same subject. The percentage marks obtained by each boy is given in table 13.1 where each boy's marks are in the same column.

Boy	Paper I (x)	Paper II (y)
1	65	78
2	73	88
3	42	60
4	52	73
5	84	92
6	60	77
7	70	84
8	79	89
9	60	70
10	83	99
11	57	73
12	77	88
13	54	70
14	66	85
15	89	89

Table 13.1

(a) Is the performance in the two papers consistent? Which is the better of the two performances?

(b) Plot the corresponding points (x, y) on a Cartesian plane and describe the resulting graph.

(c) Can you obtain a rule relating x and y? Explain your answer.

(d) Calculate the mean mark in each paper and represent the same on your graph paper. Can you describe the performance in each paper with reference to the mean mark.

(e) Can you draw a line that roughly represents the points in your graph?

(f) Calculate the median mark on each paper, denote the medians as mx and my respectively.

(g) Using the graph Table 13.1, divide the set into 3 regions each containing 5 entries.

Group I: Entries 1 to 5.

Group II: Entries 6 to 10.

Entries III: Entries 11 to 15

(i) For each group find the median marks and denote them as m_{x_1}, m_{y_1}; m_{x_2}, m_{y_2}; and m_{x_3}, m_{y_3}, respectively.

(ii) Write the medians in the coordinate form as:

$$(m_{x_1}, m_{y_1}), (m_{x_2}, m_{y_2}), (m_{x_3}, m_{y_3})$$

(iii) Plot the three points on the same graphs and join them with the best line you can draw.

Most of the statistical skills that we have developed in our earlier work were based on one variate i.e one set of data only. Such variates included heights, mass, age, marks etc. In everyday life situations, activities, circumstances etc may arise so that there is need to compare two sets of data. In this unit we shall learn how represent and analyse data that considers use of two variables for the same person or activity.

13.1 Definition and examples of bivariate data

Activity 13.1

1. Use reference books or internet to define bivariate data.
2. Give examples of bivariate data that can be obtained.
   (i) From members of your class
   (ii) From any other source.

Consider statistical data whose observations have exactly two measurement or variables for the same group of persons, items or activities.

Such data is know as Bivariate data.

Some examples of Bivariate data are

- Age (x) years and mass (y) kg
- Height (x) cm and age (y) years.
- Height (x) cm and mass (y) kg.
- Expenditure in a business (x) and profit (y) for a particular period of time etc.

13.2 Frequency distribution table for bivariate data

Activity 13.2

1. Assign numbers 1 to 5 to identify each of you.
2. For each of you, measure and record your height (cm) and mass (kg).
3. Record the data and present them in a table similar to table below.

Student	1	2	3	4	5
Height (cm) x					
Mass (kg) y					

Table 13.2

The two sets of data are denoted as x and y or any other two variables.

This data can be displayed in a frequency table using the information for the whole class for later use.

In a bivariate data, the x and y are considered as an **ordered part** written as (x_1, y_1), (x_2, y_2),.....(x_n, y_n). Meaning that both variables have equal number of elements, referring to the same entry.

Suppose a Mathematics examination consists of two distinct papers i.e. an algebra paper and a geometry paper. The mark scored by a candidate in the algebra paper is one variate, and that on the Geometry is another variate. In order to assess the overall performance in mathematics the examiner must consider the two corresponding marks together for each student.

Let x denote the algebra mark

And y denote the geometry mark.

And the number of students in the class be 15.

Student	x	y
1	45	57
2	93	69
3	35	44
4	56	75
5	16	25
6	50	45
7	63	74
8	70	93
9	86	84
10	65	45
11	77	55
12	38	48
13	64	78
14	52	62
15	22	29

Table 13.3

The set of marks (x, y) in table 13.3 above is an example of bivariate data.

Every set denotes the corresponding marks for each student. For example student member 1 scored 45% in algebra and 57% in geometry.

13.3 Review of data presentation using graphs

Activity 13.3

1. Write the information in Table 13.4, in the co-ordinate form.

Observa-tion	1	2	3	4	5	6	7	8	9	10
x	2	4	3	7	5	3	5	6	8	6
y	1	4	1	5	6	9	4	1	6	7

Table 13.4

2. Using an appropriate scale, mark the variable x on the horizontal axis, and the variable y on the vertical axis.
3. Plot the points.
4. What kind of graph do you obtain? Would it make sense to join these points?

To be able to represent any two variables graphically, one should first write the given information in co-ordinate form.

- For an accurate graph, an appropriate scale should be chosen and used.
- In order to analyse a graph accurately, it must be accurately drawn and points joined either using:
  i) A straight line or
  ii) A curve or
  iii) A series of zigzag line segments.

Example 13.1

Using tables 13.5, choose an appropriate sale and represent the information in a graph.

x	0	1	2	3	4	5	6	7
y	3	6	12	24	48	96	192	384

Table 13.5

(a) *Plot the points and join the points appropriately.*

(b) *From your graph find:*

i) *y when $x = 2.5$*

ii) *x when $y = 150$*

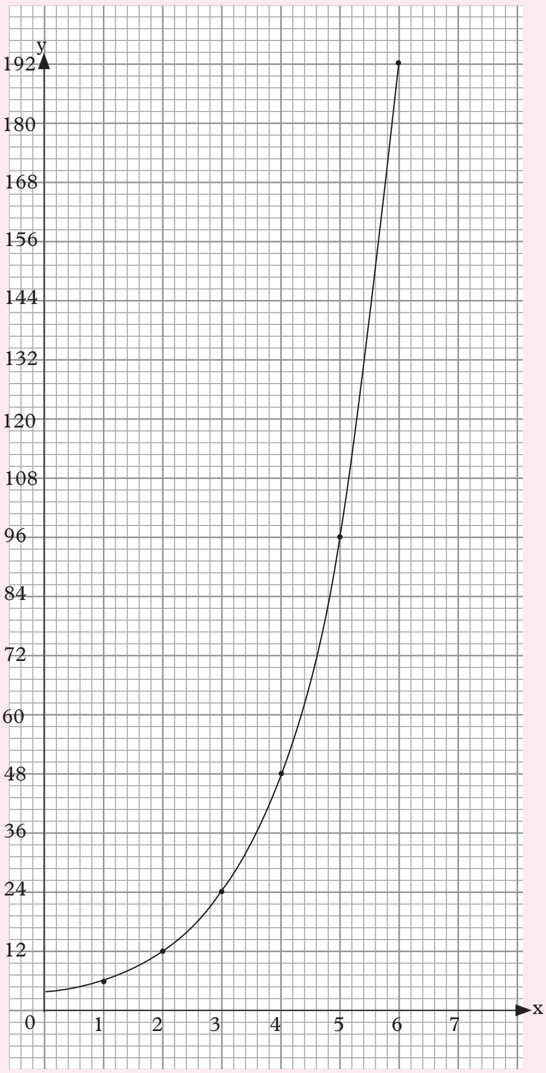
(c) *By observing the trend of your graph, estimate the value of x when $y = 500$.*

Solution

The points have co-ordinates (0, 3) (1, 6) (2, 12), (3, 24), (4, 45), (5, 96), (6, 192), (7, 384)

V scale 1 cm → 20 units

H scale 1 cm → 1 unit



Graph. 13.1

This graph represents a curve with an increasing upward trend.

(a) *Graph 13.1 shows the required graph of y against x.*

(b) *(i) when $x = 2.5, y = 30$*

(ii) when $y = 150, x = 4.6$

(c) *When $y = 500, x = 7$.*

Note:

The graph in our example results in a curve.

Exercise 13.1

Plot points for the data given in the following tables.

1.

x	10	4	2	5	7	3	8	8
y	24	12	6	15	16	10	19	21

Table 13.6

2.

x	y
155	8
158	7.5
160	7
163	8
165	8.5
168	9
170	9.5
173	8
175	9.5
178	10.5
180	11

Table 13.7

3.

x	3.00	4.10	5.60	6.00	7.00
y	15.00	10.88	9.30	3.55	4.60

Table 13.8

4. Draw the graphs for the data in the given ordered pairs (x, y).

(0.0, 10.1), (1.0, 11.2) (2.0, 12.3) (3.0, 13.4) (4.0, 14.6) (5.0, 15.9)

13.4 Representing bivariate using scatter diagrams

13.4.1 Definition of scatter diagrams

Activity 13.4

In a certain county, data relating 18 to 25 year old drivers and fatal traffic accidents was gathered and recorded over a period of 3 months. The data was presented as in table 13.9.

1. Using a suitable scale, mark age (x) years on the horizontal axis and fatal accidents (y) on the vertical axis, using the data in table 13.9.

Months 1		Month 2		Month 3	
Age (x)	Fatal Accidents (y)	Age (x)	Fatal accidents (y)	Age (x)	Fatal accidents (y)
18	8	18	10	18	6
19	7	19	7	19	8
20	10	20	7	20	4
21	5	21	6	21	7
22	6	22	4	22	5
23	4	23	6	23	5
24	2	24	2	24	4
25	3	25	3	25	2

Table 13.9

2. Plot the points (18, 8) (18,10) (18,6),…………(25, 2)
3. What patterns do the points reveal that can help you draw any conclusion?
4. Do you think any group or groups of people would be interested in the conclusion of this activity? Explain.

Often graphs are in form of straight lines, or curves or continuous jagged line segments.

However, some graphs do not fit into any of these categories. Some graphs result in a scatter of points rather than a collection of points falling on a well-defined line or curve. Sometimes there may be an underlying linear relation between the variables, but the points in the graph are scattered along it rather than fall on it. Such a graph is called a scatter graph or scatter diagram.

13.4.2 The line of best fit in a scatter diagram

Activity 13.5

Using the graph you obtained in activity 13.4, draw a line as follows.

1. Use a transparent ruler.
2. Place the ruler on the scatter diagram in the direction of the trend of the points so that you can see all the points.
3. Adjust the ruler until it appears as though it passes through the centre of the data (points).

Then draw the line.

Although the points in Fig 13.1 do not fall on a line, they seem to scatter in a specific direction. Therefore, a line that closely approximates the data can be drawn and used to analyse the data further.

In a scatter diagram, it is not easy to define a relation between the two variables involved. However, the points may appear to point some direction which may be approximated by a line known as the line of best fit.

Using points on such a line, we can analyse the given data as follows;

i) Describe the relation between the two variables.
ii) Find the equation of the line,
iii) Interpret the data i.e. read values from the graph,.
iv) Describe the trend of the graph. written R^2(A).

Example 13.2

The amount of grant allocated to 12 education institutions in a certain country in a year is listed together with the population sizes.

Population (x) (Tens of thousands)	*Grant (y) (million)*
29	8
58	17
108	34
34	10
115	34
19	7
136	41
33	10
25	9
47	13
49	17
33	13

Table 13.10

(a) *Write the data in co-ordinate form.*

(b) *Plot the points to obtain a scatter diagram.*

(c) *Draw the line of best fit.*

(d) *Identify three points on the line and use them to find the equation of the line.*

(e) *Describe the trend of the graph.*

(f) *Use the graph to estimate: (i) x when y = 42.0 ; (ii) y when x = 85.*

Solution

(a) *(29, 8) (58, 17) (108, 34) (34, 10) (115, 34) (19, 7) (136, 41) (33, 10) (25, 9) (47, 13) (49, 17) (33, 13)*

(b) *Fig 13.2 shows the required scatter diagram and the bets line of fit.*

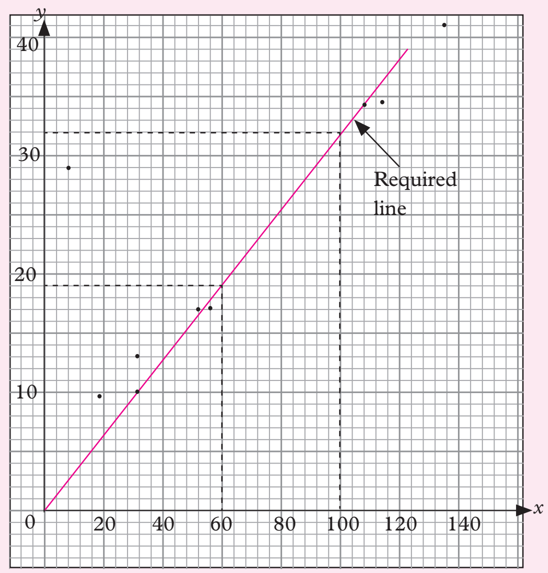


Fig. 13.2

Some points on the line (30, 10) (70, 23) (90, 30) using (30, 10) and (90, 30).

Gradient $= \frac{30 - 10}{90 - 30} = \frac{20}{60}$

$y = mx + c,\ m = \frac{20}{60} = \frac{1}{3},\ c = 0$

$y = \frac{1}{3}x$

(e) *Variables x and y are direct proportion*

(f) *When y = 42.0, x = 126*

Exercise 13.2

In this exercise use graph or squared paper where necessary.

1. Use the data in Table 13.11 to draw a scatter diagram. Use the scatter diagram obtained to estimate and draw the line of best fit.

x	10	4	2	5	7	3	8	8
y	24	12	6	15	16	10	19	21

Table 13.11

2. Table 13.12 below shows the average masses of a group of boys in the age group 5 to 14 years.

Age (years) (x)	5	6	7	8	9	10	11	12	13	14
Mass (Kg) (y)	24	25	27	28	31	31	28	41	47	55

Table 13.12

(a) Plot the points to obtain a scatter diagram.

(b) Use the scatter diagram obtained above to draw the line of best fit and describe its gradient or slope. Find its equation.

3. Table 13.13 below shows the heights (cm) and the corresponding shoes sizes for a group of people.

Height (x)	Shoe size (y)
155	8
158	7
160	7
163	8
165	8
168	9
170	9
173	8
175	9
178	10
180	10

Table 13.13

Use the data to draw a scattered diagram. Use the scatter diagram obtained to draw the line of best fit.

Use your graph to estimate the shoe size you expect someone 171 cm tall to wear.

4. From an experiment, different masses are attached at the end of a spring wire and the length of the wire noted and recorded as in Table 13.14 below.

Load (N) (x)	0.0	1.0	2.0	3.0	4.0	5.0
Length cm (y)	10.1	11.2	12.3	13.4	14.6	15.9

Table 13.14

(a) Obtain the scatter diagram of the data Table 13.4 to draw the line of best fit. Find its equation.

(b) What load might produce a spring length of 18.3 cm?

5. The data in table 13.15 was obtained from an electrical current.

Volts (x)	3.0	4.1	5.6	6.0	7.0
Velocity (y)	15.0	10.88	9.3	3.55	3.60

Table 13.15

(a) Obtain the scatter diagram for the data in table 13.15 to estimate the line of best fit.

(b) What velocity corresponds to 8.0 volts?

(c) What can you say about the gradient of the line?

(d) Find the equation of the line

Alternative method of obtaining the line of best fit (The median fit line)

Activity 13.6

Table 13.16 shows the result of a survey done on nine supermarket stores. It shows the amount of money spent on advertising each day and the corresponding amount of money earned as profit.

Store	A	B	C	D	E	F	G	H	I
Advertising (x)	30	15	17	7	32	9	23	26	36
Profit (y)	4640	1740	3250	880	5570	2090	3360	4990	5680

Table 13.16

use Table 13.17 to draw a scatter diagram.

Step 1

1. Divide the scatter diagram into three region. Each region should have the same number of points.

Step 2

1. For each region, using the table of values above,
   - (i) find the median of the x co-ordinates (x - median)
   - (ii) find the median of the y co-ordinates (y - median).
   - (iii) Write the x- and y-medians for each region in the co-ordinate form.
   - (iv) On the scatter diagram, plot the three median points.
   - (v) Place the edge of a ruler between the first and third median points. If the middle point is not on the line formed, then slide the ruler about a third of the way towards the second points without changing the direction of the shape of the line. Draw the median fit line.

Example 13.3

Table 13.17 shows the result of an experiment.

W	10	30	50	60	80	100
E	8.9	19.1	29	33	45	54

Table 13.17

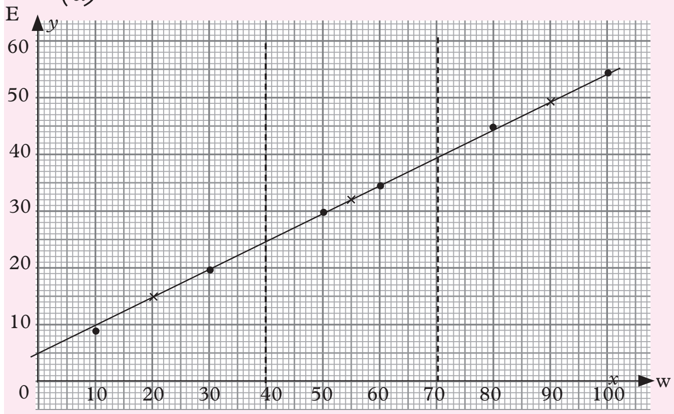
(a) Represent the data graphically.

(b) Find theme median points.

(c) Draw the line of the best fit.

Solution

(a)



Graph. 13.3

(b) The median points are (20, 14), (55, 31), and (90, 49.5). They are marked with (x) crosses on the graph.

Example 13.4

Table 13.18 shows marks obtained by students in Maths (x) and Physics (y).

Student	A	B	C	D	E	F	G	H	I	J	K
Mathematics mark (x)	41	37	38	39	49	47	42	34	36	48	29
Physics mark (y)	36	20	31	24	37	35	42	26	27	29	23

Table 13.18

(a) Plot a scatter diagram.

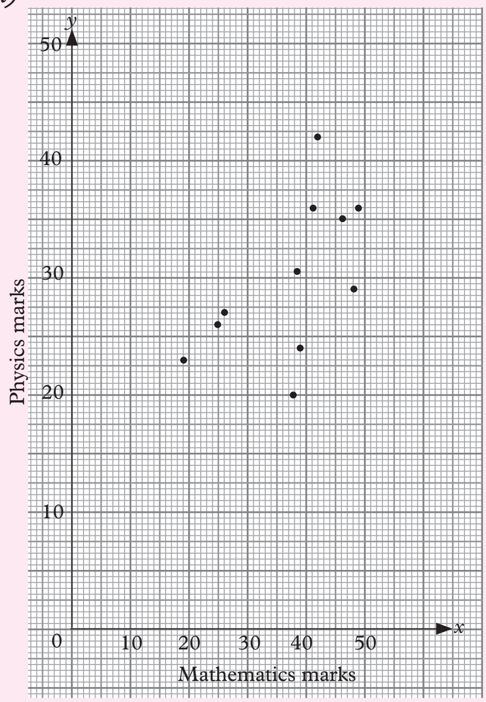
(b) Find $\bar{x}$ and $\bar{y}$.

(c) It is easy to locate the line of the best fit? Give reasons.

(d) Is there relation between performance of two subjects?

Solution

(a)



Graph. 13.4 (a)

(b)

$\bar{x} = \frac{\Sigma x}{\eta}$

$\bar{x} = \frac{41+37+38+39+49+47+42+34+36+48+29}{11}$

$\bar{x} = \frac{440}{11} = 40$ marks

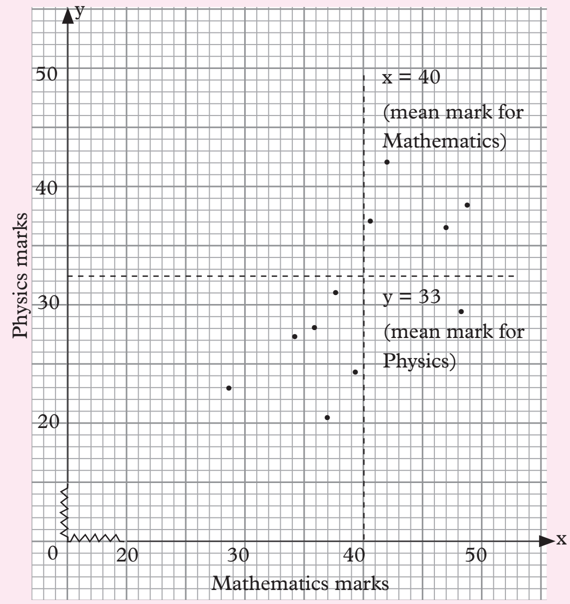
$\bar{y} = \frac{\Sigma y}{\eta}$

$\bar{y} = \frac{36+20+31+24+37+35+42+26+27+23}{11}$

$\bar{y} = \frac{440}{11} = 40$ marks

(c) It is not easy to draw the line of best fit since all the points on the scatter diagram are in different directions.

(d)



Graph 13.4 (b).

- The broken lines in Graph 13.4 (b) represent the mean marks for the two subjects.
- From this graph we can see that any Physics mark above 33 is above average score. Similarly any Mathematics mark above 40 is above average score.
- We can see that the above-average scorers are same in the two subjects.

Also the below-average scorers are the same.

From these observations, we can conclude that, for this sample, ability in Mathematics is closely associated with ability in Physics.

Exercise 13.3

1. The mass (kg) and the average daily food consumption for a group of 13 teenage boys was recorded as in Table 13.19.

Mass (kg) x	Food consumption: g 100 calories per day
84	32
93	33
81	33
61	24
95	39
86	32
90	34
78	28
85	33
72	27
65	26
79	29

Table 13.19

Use the table to:

(a) Draw a scatter diagram.

(b) Estimate the line of best fit.

(c) Describe the slope of the line.

2. Table 13.20 below shows marks scored by 25 students in two subjects, Chemistry and Social studies.

Chemistry (x)	Social studies (y)
4	54
22	17
4	10
45	34
22	26
34	25
27	21
43	37
19	50
39	24
12	43
5	16
5	27
27	38
17	7
26	25
27	28
36	34
19	13
21	56
35	35
7	25
33	51
25	57
24	12
25	57
17	7
26	25
27	28
36	34

Table 13.20

(a) Use the data to draw a scatter diagram.

(b) Use the method of mean marks to establish the type of relationship if any between the Chemistry marks and the Social studies marks.

13.5 Correlation

Activity 13.7

1. Use the dictionary or internet to find the meaning of the world correlation.

   Table 13.21 shows entrance test marks (x) and the corresponding course grade (y) for a particular year.

Entrance test score (x)	33	91	79	68	94	73	65	77	84	80
Course grade (y)	92	98	78	71	100	72	69	80	90	85

Table 13.21

2. Draw a scatter diagram for the data in table 13.21 and draw a line of best fit. Comment on the suitability of the entrance test to the placement of the students in the respective courses the co-ordinates of the image.

By definition, correlation is a mutual relationship between two or more things.

Examples of correlation in real life include:

- As students study time increases, the tests average increases too.
- As the number of trees cut down increases, soil erosion increases too.
- The more you exercise your muscles, the stronger they get.
- As a child grows, so does the clothing size.
- The more one smokes, the fewer years he will have to live.

- A student who has many absences has a decrease in grades scored.

Types of correlation

Activity 13.8

Identify positive and negative correlation situation in the following cases:

a) The more times people have unprotected sex with different partners, the more the rates of HIV in a society.

b) The more people save their incomes, the more financially stable they become.

c) As weather gets colder, air conditioning costs decrease.

d) The more alcohol are consumes, the less the judgment one has.

e) The more one cleans the house, the less likely are to be pests problems.

Correlation can be negative or positive depending on the situation:

In a situation where one variable positively affects another variable, we say **positive correlation** has occurred.

Also, when one variable affects another variable negatively, we say **negative correlation** has occurred.

Correlation is a scatter diagram which can be determined whether it is positive or negative by following the trend of the points and the gradient of the line of the best fit.

- If the gradient is positive, the positive correlation occurs.
- If the gradient is negative, then negative correlation occurs
- If gradient is zero, the there is no correlation between two variables.

Example 13.5

The amount of government bursaries allocated to certain administration regions in the country in a certain year is listed together with their population sizes.

Region	*Population (10,000s) (x)*	*Bursaries in millions (y)*
1	29	8.0
2	58	16.8
3	108	33.9
4	34	10
5	115	34
6	19	6.5
7	136	40.5
8	33	10.2
9	25	8.8
10	47	12.5
11	49	17.3
12	33	12.6

Table 13.22

Use the data in table 13.22 to draw a scatter diagram and assess whether there appears to be correlation between the two measurements labeled x and y.

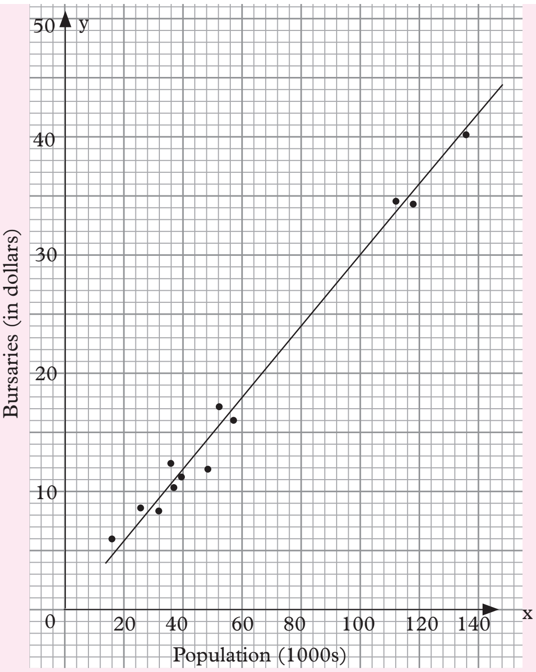
Vs: 1 cm: 50 000

Hs: 1 cm: 10 m

Solution

Graph 13.5 shows the scatter diagram and the line of bet fit of the sets of data.

The two measurements (x) and (y) have a correlation.



Graph. 13.5

*The best line of fit in this scatter diagram follows the general trend of the points. This line has a positive gradient. Thus, the relation in this data is called a **positive correlation**.*
image cordinator A″B″C″*and* D″

Example 13.6

The following measurements were made and the data recorded to the nearest cm.

Height of boys (cm)	*Height of his father cm*
164	171
168	186
150	164
162	180
159	176
165	177
187	192
152	167
180	189
166	180

Table 13.23

Use the table to:

(a) Plot the data using co-ordinate axes.

(b) Draw the line of best fit for the data

(c) Find the equation of the line in (b) above.

(d) Describe the correlation.

(e) Would it be reasonable to use your graph to estimate the height of a father whose son is 158 cm tall?

Solution

Let the height of the boy be x cm and that of the father y cm. Using a scale of 1cm – 5cm on both axes, mark x on the horizontal axis and y on the vertical axis.

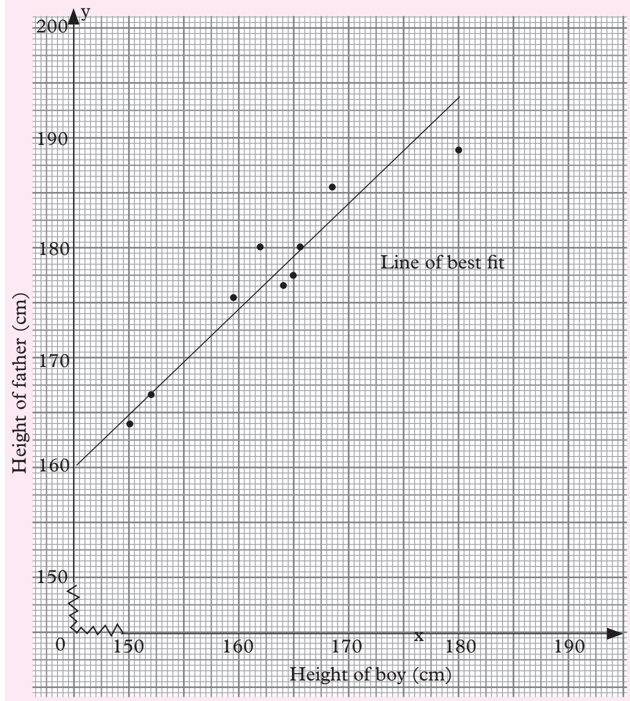
LQ = 159 ↓   m = 164.5 ↓   UQ = 168 ↓

(*x*)	150	152	159	162	164	165	166	168	180	187
(*y*)	164	167	171	176	177	180	180	186	189	192

↑ LQ = 171   ↑ m = 178.5   ↑ UQ = 186

Table 13.24

Fig 13.6 shows the required scatter diagram and line of best fit.



Graph. 13.6

Using points (152,167) and (180, 189):

$$\text{Gradient of line} = \frac{189-167}{180-152} = \frac{22}{28}$$

$$= \frac{11}{14}$$

$y = mx + c$

From the Graph C = 160

$\therefore \quad y = \frac{11}{14}x + 160$

$\therefore$ *Equation of the line of best fit can be written as* $14y = 11x + 2240$.

(d) *The correlation in this data is positive or direct. The line of best fit has a positive gradient and therefore a positive trend.*

(e) *Yes*

Height of father is 175 cm.

Exercise 13.4

1. The set of data in Table 13.24 shows number of vehicles and road deaths in some 10 countries.

Country	1	2	3	4	5	6	7	9	9	10
Vehicles per 100 population (*x*)	31	32	30	47	30	19	36	40	47	58
Road deaths per 100000 population (*y*)	14	29	22	32	24	20	21	22	30	35

Table 13.24

Use table 13.24 to draw a scatter diagram and use it to determine whether there is a correlation between x and y. If yes, describe the correlation.

2. Data were collected on the mass of a rabbit in kilograms at various ages in weeks. Table 13.25

Age	1	2	3	4	5	6	7
Mass	0.8	1.1	1.2	1.4	1.5	1.5	1.7

Table 13.25

Use the data in the table above to:

(a) Draw a scatter diagram for the data

(b) Draw a line of best fit

(c) Use your graph to predict the mass of the rabbit when it is 8 weeks.

(d) How would you describe the correlation in this data?

3. From a laboratory research, data were collected on mass of hen and mass of the heart of hen and recorded table 13.26

Mass of rabbit (kg)	1.22	1.54	1.26	1.19	1.23
Mass of heart (mg)	772	837	761	910	691

Table 13.26

Use the table to:

(a) Construct a scatter diagram for the data.

(b) Estimate a line of best fit.

(c) Find the equation of the line.

(d) How can you use the equation in c to make predictions?

(e) Describe the correlation in this data.

4. In a junior cross country race the masses (kg) of sample participants were checked and recorded as they entered the race in Table 13.27. Their finishing positions in the race were also noted. Use a scatter diagram to determine whether there was any correlation between the sets of data. Give reason(s) for your answer.

Mass (x)	Finishing position (y)
60	2
63	6
70	2
65	4
60	6
64	5
67	4
73	2
56	3
58	2
60	1
60	3
66	3
65	3
61	1
60	2
64	3
53	1
68	3
65	2
55	2
60	3
60	7
65	1

Table 13.27

5. A basketball coach recorded the amount of time each player played and the number of points the player scored. Table 13.28 shows the data.

Time(minutes) (x)	Points scored (y)
151	51
18	10
164	38
86	28
136	39
110	40
55	8
163	62
192	46
98	32
71	25

Table 13.28

(a) Make a scatter diagram and estimate the line of best fit.
(b) Describe the type of correlation if any.
(c) Use your graph to estimate how much time a player who scored 67 points played.

6. The production manager had 10 newly recruited workers under him. For one week, he kept a record of the number of times that each employee needed help with a task, table 13.29.

Employee	Length of employment (wks)	Request for help
A	24	14
B	11	20
C	47	10
D	58	13
E	3	25
F	70	16
G	76	6
H	44	15
I	33	19
J	87	6

Table 13.29

(a) Make a scatter diagram for the data and estimate the line of best fit.
(b) What type of correlation is there?
(c) What conclusion do you make from your graph? Justify your answer.

Unit Summary

The data in the Table 13.30 belows belong to the same sample.

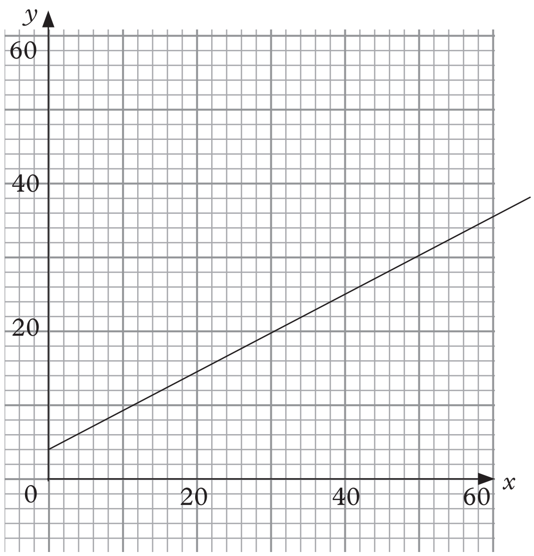
x	46	17	32	6	55	21	33	41	50	56
y	33	17	19	7	34	9	25	29	28	39

Table 13.30

- Such data is called Bivariate data. This data has two variants x and y.
  - The data has ten entries

- Each entry has two variants x and y
- This data can be presented graphically using co-ordinates. (x,y) ie. (46, 33) (17, 17)........... (56, 39)

- The graph below represents the given data. Such a graph is called a scatter diagram. The point do not lie on a line or a curve, hence the name.



Graph. 13.7

- The line drawn through the points is an approximation. It is called the line of best fit. It shows the general trend of the data or graph.
- The equation of such a line can be found and used to analyse data equation: $20y = 11x + 80$.
- The line of best fit shows that there is a relationship between the two data sets though the line is an approximation. If it shows a positive trend i.e. the line has a positive gradient thus we say the data has a correlation. Since the line has a positive gradient, the correlation is positive.
  If the line had a negative gradient, we would say the correlation is negative.
- We can use the graph to estimate missing variants. For example, we can find:
  (i) y when $x = 29$; Ans $y = 20$
  (ii) x when $y = 3$; Ans $x = 60$
  (iii) y when $x = 44$; Ans $y = 28$

Unit 13 Test

1. The table below shows the data collected and recorded on ten football players in a season.

Points earned (x)	Number of goals (y)
40	35
15	18
28	20
5	8
48	37
18	10
29	27
36	31
43	30
49	42

Table 13.31

Use the given information to answer the following questions.

(a) Make a scatter diagram for the data.

(b) Estimate the line of best fit representing the data in (Table 13.31).

(c) Use your graph to estimate:

  (i) The number of points earned by a player who scored 25 goals.

  (ii) The number of goals scored by a player who earned 45 points.

(d) Explain how for any line, you know whether a slope (gradient) is positive or negative.

(e) Find the equation of the line you drew in question (b) above.

(f) Describe the correlation in the data in this question.

(g) What conclusion can you draw from this graph?

2. Data below was collected from a certain supermarket in Kigali City. The price is in Francs:

Commodity	Price in May 2016	Price in June 2016
A	12	18
B	15	24
C	30	36
D	21	42
E	9	18
F	27	36

Table 13.32

(a) Calculate the average price for all commodities.

(i) May 2016

(ii) June 2016

(b) Plot a scatter diagram for the two prices for all the commodities.

(c) Draw the line of best fit for the data.

(d) What conclusion can you draw from the scatter diagram plotted?

3. (a) Calculate the $\bar{x}$ and $\bar{y}$ from the data given below:

x	5	7	9	12
y	6	5	3	6

Table 13.33

(b) Plot a scatter diagram for the data above.

4. The table below shows the number of students (x) and the number of days (y) they remained at school at the end of term one in 2017.

x	y
5	12
6	14
7	16
8	25
9	9
10	8
11	7
12	4

(a) Make a scatter diagram for the data.

(b) Explain how any line of best fit can be draw.

(c) Describe the correlation of the data.

Glossary

1. ***Altitude theorem*** – the altitude to the hypotenuse to a right-angled triangle is the mean proportional between the segments it divides the hypotenuse.
2. ***Chord*** **-** is a straight line joining any two points on the circumference points of a circle.
3. ***Collinearlity*** - three or more points are said to be collinear if they lie on a single straight line.
4. ***Compound interest formula*** **-** $A = P(1 + r/100)^n$ where ***n*** is the number of interest periods, ***A*** is the accumulated amount, ***r*** is the rate and ***P*** is the principal.
5. ***Compound interest*** is the interest calculated on the initial principal and also on the accumulated interest of previous periods of a deposit or loan.
6. ***Digit*** – a digit is any numeral from 0 to 9.
7. **Enlargement** - is a kind of transformation that changes the size of an object, it includes making objects smaller because the shape can be bigger or smaller according to the scale factor, *k*.
8. ***Inequality*** - is a statement that involves a comparison between at least two quantities or a set of values.
9. ***Leg theorem*** – the leg of a right-angled triangle is the mean proportional between the hypotenuse and the projection of the leg of the hypotenuse.
10. ***Linear function*** - is any expression of the form $y = mx + c$ where m = gradient and $c = y$ - intercept.
11. ***Median theorem*** states that the median from the right-angled vertex to the hypotenuse is half the length of the hypotenuse.
12. ***Number*** - a number is an idea; a ***numeral*** is the symbol that represents the number.
13. ***Orthogonal vectors*** - two vectors are said to be orthogonal if the angle between them is 90°/perpendicular (i.e. if they form a right angle).
14. ***Pythagoras theorem*** - $a^2 + b^2 = c^2$
15. ***Quadratic equation*** is an equation that is of the form $ax^2 + bx + c = 0$ where *a*, *b* and *c* are constants and $a \neq 0$.
16. ***Rational equations*** are algebraic equations involving fractions.
17. ***Reflection*** is a trans-formation representing a flip of a figure.
18. ***Reverse percentage*** involves working out the original price of a product backwards after the increase of its price.
19. ***Rotation*** is the turning of an object about a fixed point or axis**.** The amount of turning is called the ***angle of rotation***.
20. ***Secant*** – is a line that intersects a circle at two points.
21. ***Sector*** – Is the part of the circle that is enclosed by two radii of a circle and their interrupted arc.

22. ***Set*** -is a collection of distinct objects, considered as an object in its own right.
23. ***Similarity*** - two triangles are similar if the corresponding sides are in proportion, i.e. have a constant ratio or the corresponding angles are equal.
24. ***Tangent*** – Is a line touching the circumference of a circle at only one point and does not cut the circumference.
25. ***Unwanted region*** is the region in which the inequality is not satisfied and is never shaded.
26. ***Venn diagram*** – is a diagram representing mathematical or logical sets pictorially as circles or closed curves within an enclosing rectangle (the universal set), common elements of the sets being represented by intersections of the circles.

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